D KNOWLEDGE RECAP: ORDINARY DIFFERENTIAL EQUATIONS (ODE)

ODEs are equations that involve ordinary derivatives. This recap section is made to be example-based such that the reader can quickly recall the techniques they have learned.

The 1st example

$$\frac{dx}{dy} = \cos(t)$$

This is perhaps the most straight forward one to solve, we can easily get:

$$x = \sin(t) + C\,, \quad C \text{ is a constant}\,.$$

Of course if you are dealing with:

$$\frac{dx}{dy} = 0$$

then,

$$x = C\,, \quad C \text{ is a constant}\,.$$

The 2nd example

$$\frac{dx}{dt} = \sin(t) + t^2$$

To find $x(t)$, we first integrate both sides of the equation:

$$\int \frac{dx}{dt}\,dt = \int \sin(t) + t^2\,dt$$

We can also discover $x(t)$ by straight forward computation of the indefinite integral:

$$x = -\cos(t) + \frac{t^3}{3} + C$$

C is a constant, although from the differentiation's point of view, a constant is not interesting at all but in many application scenarios we still would like to know the constant. We need initial conditions, for instance $x(t_0) = K$. Then,

$$\int_{t_0}^{t} \frac{dx}{d\tau}\,d\tau = \int_{t_0}^{t} \sin\,\tau + \tau^2\,d\tau$$

We may find:

$$x(t) = -\cos\,t + \frac{t^3}{3} + \cos\,t_0 - \frac{t_0^3}{3} + K$$

The 3rd example

$$\frac{\mathrm{d}\,x(t)}{\mathrm{d}\,t} = mx(t) + n\,, \quad \text{with } m \neq 0\,.$$

We can find $x(t)$ by separating variables to two sides of the equation and integrate both sides.

$$\frac{\frac{\mathrm{d}\,x(t)}{\mathrm{d}\,t}}{mx(t)+n} = 1$$

$$\frac{\mathrm{d}\,x(t)}{mx(t)+n} = 1\,\mathrm{d}\,t$$

We completed the separation and now we integrate both sides to find $x(t)$,

$$\int \frac{1}{mx(t)+n}\,\mathrm{d}\,x = \int 1\,\mathrm{d}\,t$$

$$\frac{1}{m}\log|mx(t)+n| + C_l = t + C_r$$

$$|mx(t)+n| = e^{mt}e^{mC_r - mC_l}$$

$$x(t) = \frac{e^{mC_r - mC_l}}{m}e^{mt} - \frac{n}{m}$$

$$x(t) = Ce^{mt} - \frac{n}{m}$$

The 4th example

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} + x(t) = t^2$$

We see that there is already a separation of variables, we attempt to integrate both sides,

$$\int \frac{\mathrm{d}\,x}{\mathrm{d}\,t} + x(t)\,\mathrm{d}\,t = \int t^2\,\mathrm{d}\,t$$

The right part of the equation is easily approachable but the left part of the equation looks terrible! When we recall the Lebniz product rule for derivatives:

$$\frac{\mathrm{d}\,xy}{\mathrm{d}\,t} = \frac{\mathrm{d}\,x}{\mathrm{d}\,t}y(t) + x(t)\frac{\mathrm{d}\,x}{\mathrm{d}\,t}$$

If there is a function $\gamma(t)$ such that:

$$\frac{\mathrm{d}\,\gamma(t)}{\mathrm{d}\,t} = \gamma(t)\,.$$

Then we could utilize this property of $\gamma(t)$ and multiply it to both sides of our ODE, such that on the left hand side of the equation we may construct the right hand side of the product rule:

$$\frac{\mathrm{d}\,x\gamma}{\mathrm{d}\,t} = \frac{\mathrm{d}\,x}{\mathrm{d}\,t}\gamma(t) + x(t)\frac{\mathrm{d}\,\gamma}{\mathrm{d}\,t} = \gamma(t)t^2$$

Luckily, we have such a $\gamma(t)$:

$$\gamma(t) = \mathbf{e^t} = \frac{\mathrm{d}\,\gamma}{\mathrm{d}\,t}$$

Thus, our ODE becomes:

$$e^t \frac{dx}{dt} + x(t)e^t = e^t t^2$$

We integrate both sides now:

$$\int e^t \frac{dx}{dt} + x(t)e^t \, dt = \int e^t t^2 \, dt$$

Using the Lebniz product rule for the left hand side and utilize the integration by parts formula

$$\int x(t)y'(t) \, dt = x(t)y(t) - \int x'(t)y(t) \, dt$$

to the right hand side (twice), we will come to:

$$e^t x(t) = e^t(t^2 - 2t + 2) + C, \quad \text{where C is a constant.}$$

Then we find:

$$x(t) = t^2 - 2t + 2 + Ce^{-t}$$

The 5th example

$$2\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + x(t) = t$$

We again make use of use the exponential function $e^{\lambda t}$ but now with a constant factor λ on the exponent, such that:

$$\frac{d}{dt}e^{\lambda t} = \lambda e^{\lambda t} \qquad \frac{d^n}{dt^n}e^{\lambda t} = \lambda^n e^{\lambda t}$$

We assume $x(t) = e^{\lambda t}$, put this in to the original ODE, we obtain the homogeneous part:

$$2\lambda^2 e^{\lambda t} + 3\lambda e^{\lambda t} + e^{\lambda t} = 0$$

The characteristic equation is the quadratic polynomial:

$$2\lambda^2 + 3\lambda + 1 = 0$$

The roots are:

$$\lambda_1 = -1, \quad \lambda_2 = -0.5$$

Thus the general solution is:

$$x_h(t) = C_1 e^{-t} + C_2 e^{-0.5t}$$

To find the particular solution, we try $x_p(t)$ is of the form $at + b$. Putting this into the original ODE yields:

$$2 \cdot 0 + 3a + at + b = t$$

Then we find:

$$a = 1, \quad b = -3$$

Thus the particular solution:

$$x_p(t) = t - 3$$

The solution is the sum of $x_h(t)$ and $x_p(t)$:

$$x(t) = x_h(t) + x_p(t) = C_1 e^{-t} + C_2 e^{-0.5t} + t - 3$$