UNIVERSITY OF APPLIED SCIENCES

BASIC CONTROL SYSTEMS

08 FREQUENCY RESPONSE AND STABILITY

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WHERE STUDENTS MATTER

STABILITY $L(s)$

For stability:

the poles should not go across the imaginary axis such that the real part is larger than zero!

We bring back our standard closed loop transfer function.
Characteristic equation:

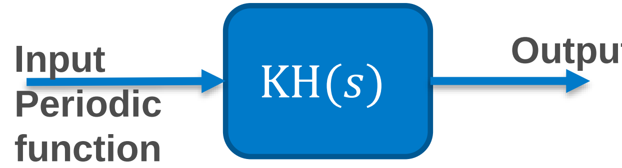
$$1 + L(s) = 0, \qquad L(s) = KG(s)H(s)$$

Poles $s = p$ should satisfy: $L(s) = -1$. The polar form:

$$|L(s)|\widehat{L(s)} = 1\, e^{\pm j\pi}$$

In the Root Locus exercise, we have looked at K, but what about frequency and phase?

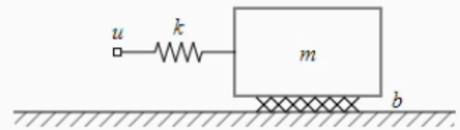
STABILITY & FREQUENCY RESPONSE

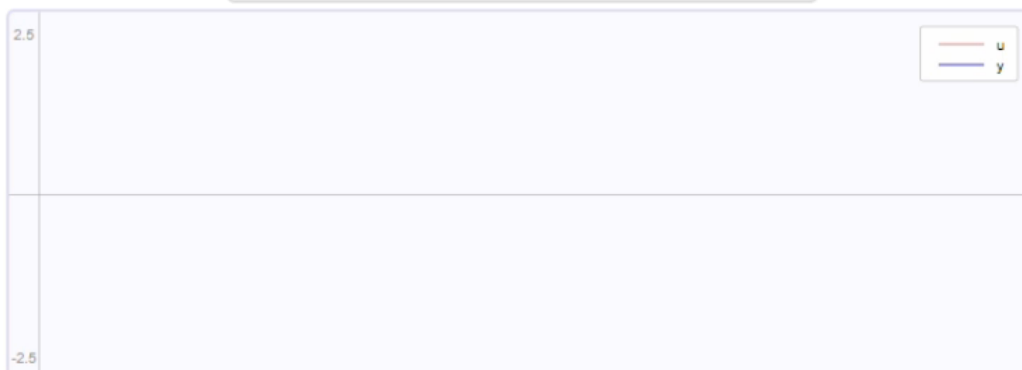


consider the system below. (Hit start button to show animation)

Click here for an animation of an analogous electrical system.

start



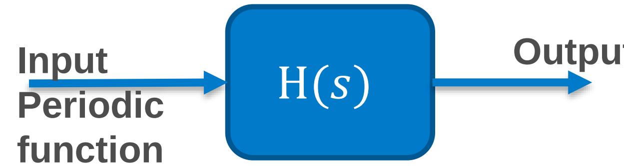


Animation by Ames Bielenberg

input to system, y=output (the position of the mass):

$$H(s) = \frac{Y(s)}{U(s)} = \frac{k}{ms^2 + bs + k} = \frac{1.6}{s^2 + 0.5s + 1.6}$$

STABILITY & FREQUENCY RESPONSE



Periodic input!

There exist a frequency ω.

Let's just assume input is $\mathrm{x(t)} = \mathrm{A}\sin(\omega t)$ as $t \geq 0$.

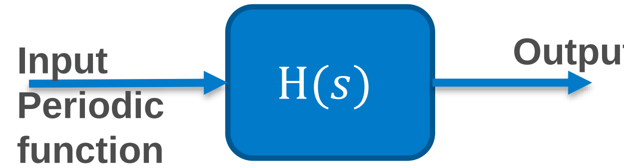
In s-domain we have output:

$$Y(s) = X(s)H(s) = \frac{A}{s^2 + \omega^2} H(s)$$

To find the frequency response, we force the real part of s: $\boldsymbol{\sigma = 0}$
And thus $\boldsymbol{s = j\omega}$.

The frequency response of the system can be discovered by $\mathbf{H}(\boldsymbol{j\omega})$.

STABILITY & FREQUENCY RESPONSE



Periodic input!

There exist a frequency ω.

Let's just assume input is $\mathrm{x(t)} = \mathrm{A}\sin(\omega_0 t)$ as $t \geq 0$.

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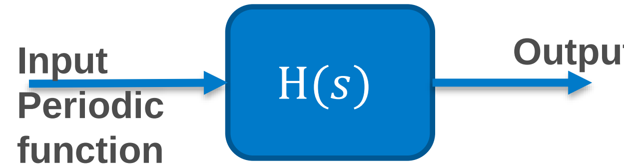
To find the frequency response, we force the real part of s: $\sigma = 0$
And thus $s = j\omega$.

The frequency response of the system can be discovered by $H(j\omega)$.

But can we do this???

STABILITY & FREQUENCY RESPONSE



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$$Y(s) = X(s)H(s) = \frac{A}{s^2 + \omega_0^2} H(s)$$

We do partial fraction decomposition to $Y(s)$:

$$Y(s) = X(s)H(s) = \boxed{\frac{M}{s + j\omega_0} + \frac{N}{s - j\omega_0}} + \boxed{\{H(s)\ decomposition\}}$$

Steady-state (**forced**) response

Transient-state (**natural**) response

STABILITY & FREQUENCY RESPONSE

We do partial fraction decomposition to $Y(s)$:

$$Y(s) = X(s)H(s) = \boxed{\frac{M}{s+j\omega_0} + \frac{N}{s-j\omega_0}} + \{H(s)\ decomposition\}$$

Based on the uniqueness of Laurent series, M is the coefficient of $\frac{1}{s+j\omega_0}$ in the Laurent series expansion of Y(s) about the singularity point $s = -j\omega_0$.

Then we may conveniently utilize the residue theorem:

$$M = \mathrm{Res}(\mathrm{Y(s)}, -\mathrm{j}\omega_0)$$

As $s = -j\omega$ is a simple root, thus assume $H(s) = \frac{P(s)}{Q(s)}$,

$$M = \mathrm{Res}(\mathrm{Y(s)}, -\mathrm{j}\omega_0) = \frac{A \cdot P(-j\omega_0)}{\frac{\mathrm{d}\,(s^2+\omega_0^2)Q(s)}{\mathrm{d}s}\Big|_{s=-j\omega_0}}$$

STABILITY & FREQUENCY RESPONSE

We do partial fraction decomposition to $Y(s)$:

$$Y(s) = X(s)H(s) = \boxed{\frac{M}{s + j\omega_0} + \frac{N}{s - j\omega_0}} + \{H(s)\ decomposition\}$$

$$\boldsymbol{M} = \mathrm{Res}(\mathrm{Y(s)}, -\mathrm{j}\omega_0) = \frac{A \cdot P(-j\omega_0)}{\frac{\mathrm{d}\,(s^2+\omega_0^2)Q(s)}{\mathrm{d}s}\Big|_{s=-j\omega_0}} = \frac{AP(-j\omega_0)}{-2j\omega_0 Q(-j\omega_0)}$$

$$= \frac{jA}{2\omega_0} H(-j\omega_0) = \frac{jA}{2\omega_0} H(j\omega_0)$$

Similarily,

$$\boldsymbol{N} = \mathrm{Res}(\mathrm{Y(s)}, \mathrm{j}\omega_0) = \frac{A \cdot P(j\omega_0)}{\frac{\mathrm{d}\,(s^2+\omega_0^2)Q(s)}{\mathrm{d}s}\Big|_{s=j\omega_0}} = \frac{AP(j\omega_0)}{2j\omega_0 Q(j\omega_0)}$$

$$= -\frac{jA}{2\omega_0} H(j\omega_0) = \overline{\boldsymbol{M}}$$

STABILITY & FREQUENCY RESPONSE

We do partial fraction decomposition to $Y(s)$:

$$Y(s) = X(s)H(s) = \boxed{\frac{\boldsymbol{M}}{s + j\omega_0} + \frac{\bar{\boldsymbol{M}}}{s - j\omega_0}} + \{H(s)\ decomposition\}$$

We look at the forced response:

$$Y_{forced}(s) = \frac{M}{s+j\omega_0} + \frac{\bar{M}}{s-j\omega_0}, \quad M = \frac{jA}{2\omega_0} H(j\omega_0)$$

$$Y_{forced}(s) = \frac{M}{s + j\omega_0} + \frac{\bar{M}}{s - j\omega_0} = \frac{(s - j\omega_0 - s - j\omega_0)\frac{jA}{2\omega_0} H(j\omega_0)}{s^2 + \omega_0^2}$$

$$= \frac{A}{s^2 + \omega_0^2} H(j\omega_0)$$

The frequency response of the system to a periodic input of frequency ω can be found via $\boldsymbol{H(j\omega)}$

STABILITY & FREQUENCY RESPONSE

$$Y_{forced}(s) = \frac{M}{s + j\omega_0} + \frac{\bar{M}}{s - j\omega_0} = \frac{(s - j\omega_0 - s - j\omega_0)\frac{jA}{2\omega_0}H(j\omega_0)}{s^2 + \omega_0^2}$$

$$\boxed{= \frac{A}{s^2 + \omega_0^2}H(j\omega_0)}$$

Employing inverse Laplace transform:

$$y_{forced}(t) = A|H(j\omega_0)| \cdot \cos(\omega_0 + \angle H(j\omega_0))$$

$$\boldsymbol{L}[f(t)] = \int_0^\infty f(t)e^{-st}\mathrm{d}t$$

From CONVOLUTION to MULTIPLICATION

$$f(t) * g(t) = \int_0^t f(\tau)g(t - \tau)\,\mathrm{d}\tau$$

With Laplace transform:

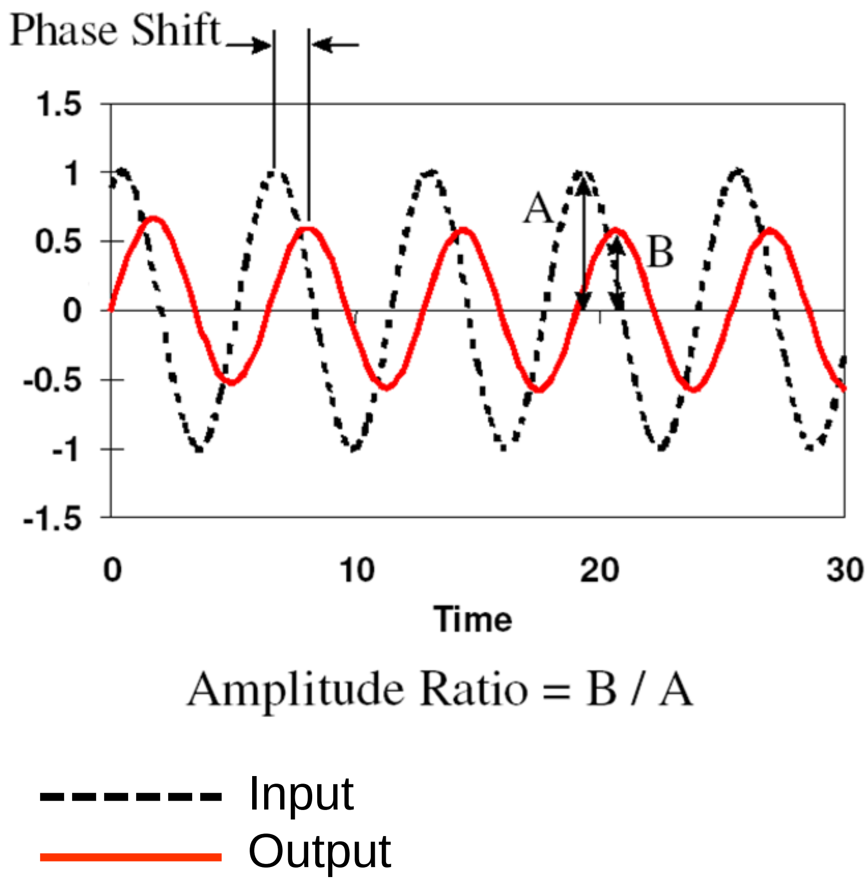
$$\boldsymbol{f}(t) * \boldsymbol{g}(t) \Leftrightarrow \mathbf{F}(s)\mathbf{G}(s)$$

Convolution in t-domain in becomes multiplication in s-domain.

By varying the input sinusoidal frequency ω_0, we may easily recover the frequency response of the system.

Frequency-response

- Frequency-response: steady-state response of systems to sinusoidal inputs



- The figure compares the output response of a system with a sinusoidal input

- Both the magnitude and the phase shift of a system will change with the frequency of the input into the system

LOGARITHMIC SCALE: DECIBELS

$$\mathrm{dB} = 20\log_{10} linear$$

$$linear = 10^{\frac{dB}{20}}$$

WHY 20 LOG_{10}

Why $20\log_{10}$?

Usually we have $\mathrm{dB} = 10\log_{10}(\frac{P_{out}}{P_{in}})$ for power measurements

In electrical circuits:

$$P = \frac{U^2}{R} = I^2R$$
$$P \sim U^2, I^2$$

We usually check voltage and current as our inputs and outputs, and that's typically what we measure. (Remember our RC low pass example)

So we have $\mathrm{dB} = 10\log_{10}(\frac{U_{out}^2}{U_{in}^2}) = 20\log_{10}(\frac{U_{out}}{U_{in}})$

$|L(s)|$

For rational functions:

$$L(s) = K_0 \frac{(s+a)(s+b)}{(s+c)(s+d)} = K_0 \frac{cd}{ab} \frac{\left(\frac{s}{a}+1\right)\left(\frac{s}{b}+1\right)}{\left(\frac{s}{c}+1\right)\left(\frac{s}{d}+1\right)}$$

$$= K_{\text{Gain}} \frac{\left(\frac{s}{a}+1\right)\left(\frac{s}{b}+1\right)}{\left(\frac{s}{c}+1\right)\left(\frac{s}{d}+1\right)}, \qquad K_{\text{Gain}} = K_0 \frac{cd}{ab}$$

Working in logarithmic allows us to transfer multiplication and division into addition and subtraction:

$$20\log_{10} K_{\text{Gain}} + 20\log_{10}\left|\frac{s}{a}+1\right| + 20\log_{10}\left|\frac{s}{b}+1\right| - 20\log_{10}\left|\frac{s}{c}+1\right|$$
$$- 20\log_{10}\left|\frac{s}{d}+1\right|$$

$|L(j\omega)|$ - THE EFFECT OF POLES AND ZEROS

For rational functions:

$$L(j\omega) = K_{\text{Gain}} \frac{\left(\frac{s}{a}+1\right)\left(\frac{s}{b}+1\right)}{\left(\frac{s}{c}+1\right)\left(\frac{s}{d}+1\right)}, \qquad K_{\text{Gain}} = K_0 \frac{cd}{ab}$$

Behavior of $z(s) = \left(\frac{s}{a}+1\right)$, with $\mathrm{s} = j\omega : |z(s)| = \sqrt{\frac{\omega^2}{a^2}+1}$

when $\omega \ll a, |z(s)| \to 1;$

when $\omega = \mathrm{a}, |z(s)| \to \sqrt{2};$

when $\omega \gg a, |z(s)| \to \infty;$

Numerator (where zeros of L(s))

Behavior of $\mathrm{p}(s) = \frac{1}{\left(\frac{s}{c}+1\right)} \xrightarrow{\mathrm{s}=j\omega} \frac{c(c-j\omega)}{\omega^2+c^2}$: $|p(s)| = \frac{c}{\omega^2+c^2}\sqrt{\omega^2+c^2}$

when $\omega \ll c, |p(s)| \to 1;$

when $\omega = c, |p(s)| \to \frac{\sqrt{2}}{2};$

when $\omega \gg c, |p(s)| \to 0;$

Denominator (where poles of L(s))

$|L(j\omega)|$ - THE EFFECT OF POLES AND ZEROS

Behavior of $z(s) = \left(\frac{s}{a} + 1\right)$, with $s = j\omega$: $|z(s)| = \sqrt{\frac{\omega^2}{a^2} + 1}$

when $\omega \ll a, |z(s)| \to 1;$

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when $\omega \ll c, |p(s)| \to 1;$

when $\omega = c, |p(s)| \to \frac{\sqrt{2}}{2};$

when $\omega \gg c, |p(s)| \to 0;$

Decreasing how fast when $\omega \gg a$?

$$\frac{1}{\sqrt{\frac{\omega^2}{a^2}+1}} \approx \frac{1}{\frac{\omega}{a}} \approx \frac{a}{\omega},$$

$$20\log_{10}\left(\frac{a}{\omega}\right)$$

$$= 20\log_{10}(a) - 20\log_{10}(\omega)$$

Rate of change: **-20 dB**

Increasing how fast when $\omega \gg a$?

$$\sqrt{\frac{\omega^2}{a^2} + 1} \approx \frac{\omega}{a} ,$$

$$20\log_{10}\left(\frac{\omega}{a}\right)$$

$$= 20\log_{10}(\omega) - 20\log_{10}(a)$$

Rate of change: **20 dB**

$\angle L(j\omega)$

For rational functions:

$$L(j\omega) = K_{\text{Gain}} \frac{\left(\frac{s}{a}+1\right)\left(\frac{s}{b}+1\right)}{\left(\frac{s}{c}+1\right)\left(\frac{s}{d}+1\right)}, \qquad K_{\text{Gain}} = K_0 \frac{cd}{ab}$$

$$\angle L(s) = \angle\left(\frac{s}{a}+1\right) + \angle\left(\frac{s}{b}+1\right) - \angle\left(\frac{s}{c}+1\right) - \angle\left(\frac{s}{d}+1\right)$$

For the phase of $z(j\omega) = 1 + j\frac{\omega}{a}$

when $\omega \ll a, \angle z(s) \to 0^o$;
when $\omega = a, \angle z(s) \to 45^o$;
when $\omega \gg a, \angle z(s) \to 90^o$;

For the phase of $\mathrm{p}(j\omega) = \frac{c^2}{\omega^2+c^2} - j\frac{\omega c}{\omega^2+c^2}$

when $\omega \ll a, \angle p(s) \to 0^o$;
when $\omega = a, \angle p(s) \to -45^o$;
when $\omega \gg a, \angle p(s) \to -90^o$;

$\angle L(j\omega)$ - THE EFFECT OF L(s) POLES AND ZEROS

For rational functions:

$$L(j\omega) = K_{\text{Gain}} \frac{\boxed{\left(\frac{s}{a}+1\right)}\left(\frac{s}{b}+1\right)}{\boxed{\left(\frac{s}{c}+1\right)}\left(\frac{s}{d}+1\right)}, \qquad K_{\text{Gain}} = K_0 \frac{cd}{ab}$$

For the phase of

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when $\omega \ll a, z(s) \rightarrow 0^o$;
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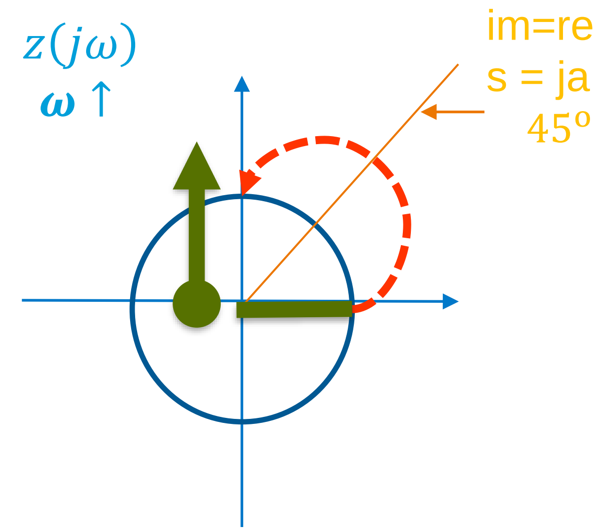

For the phase of

$\mathrm{p}(j\omega) = \frac{c^2}{\omega^2+c^2} - j\frac{\omega c}{\omega^2+c^2}$:

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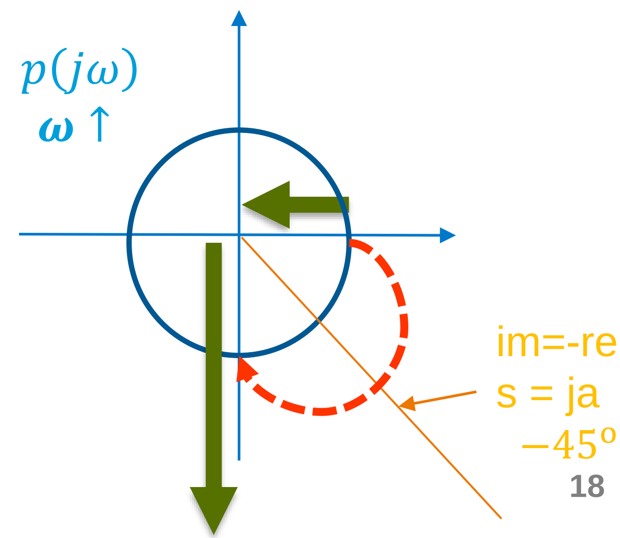
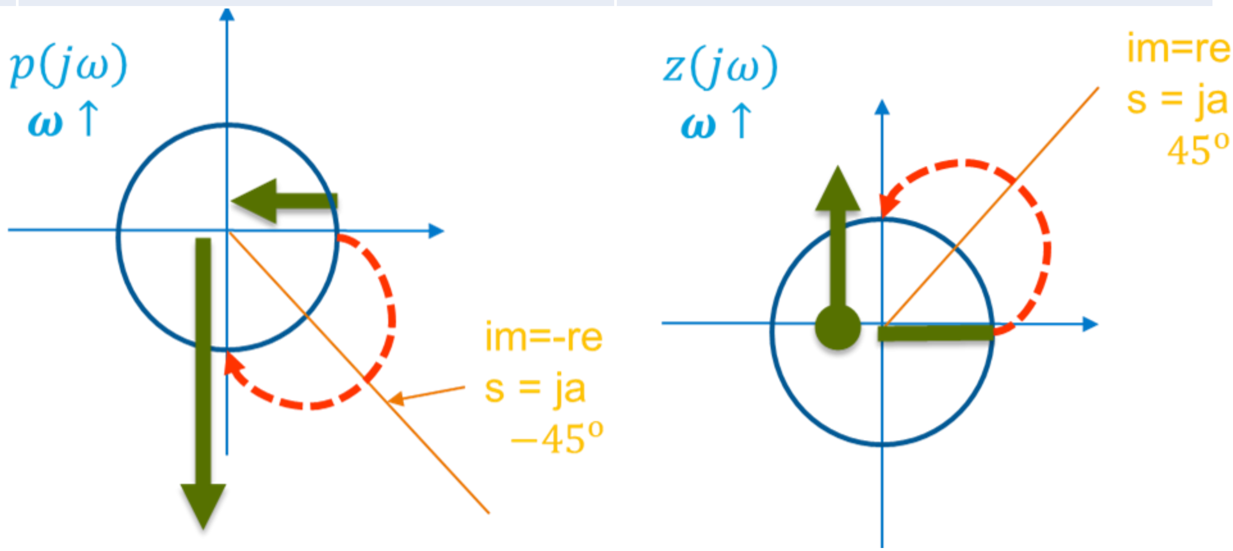


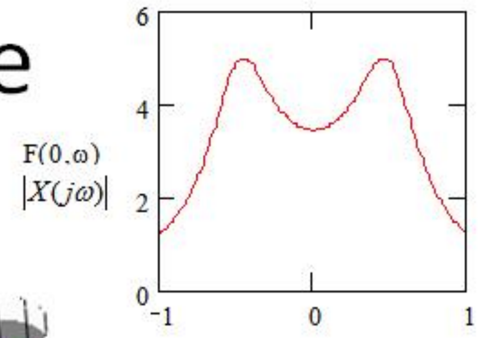
POLE ZERO FREQUENCY EFFECT

$L(s)$ Log scale	Poles of $L(s)$ $(s=-p)$ $\dfrac{1}{\frac{s}{p}+1}$	Zeros of $L(s)$ $(s=-z)$ $\frac{s}{z}+1$
Magnitude	Subtraction (Suppress $\omega > p$)	Addition (Boost $\omega > z$)
Phase	Clockwise 90^o	Counter Clockwise 90^o



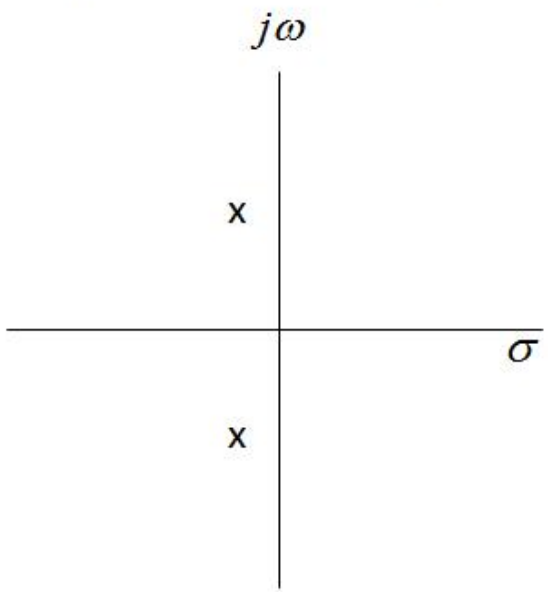
POLE ZERO FREQUENCY EFFECT

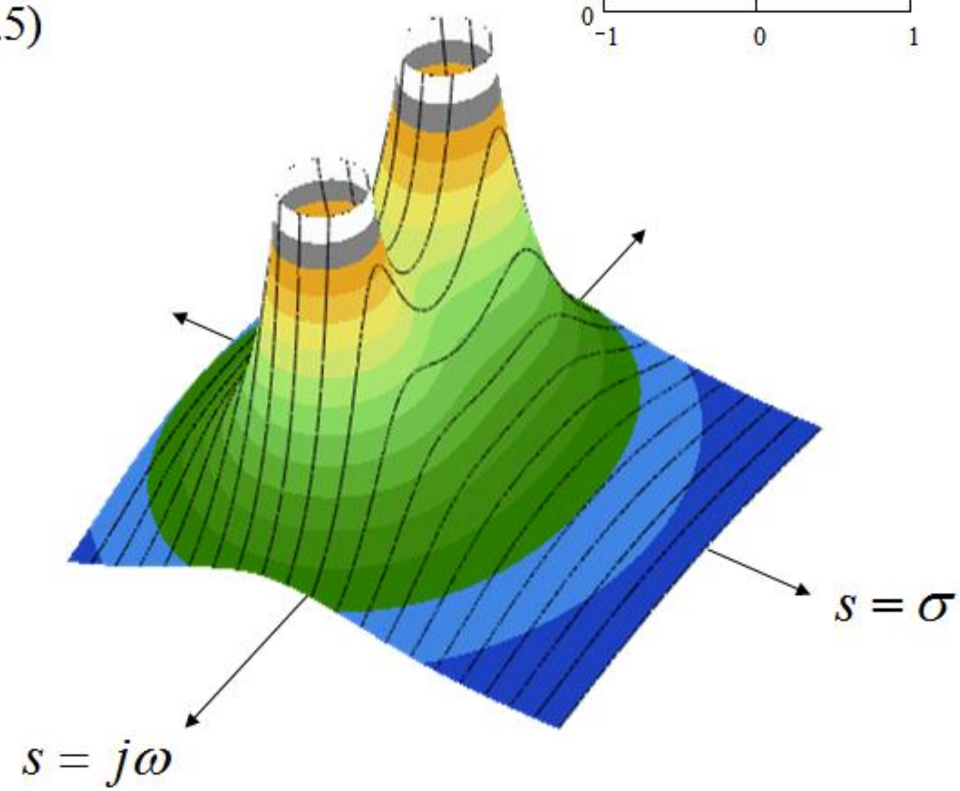
Filter Example



$$X(s)=\frac{1}{(s+.2+j.5)(s+.2-j.5)}$$

(poles at $s=-.2\pm j.5$)





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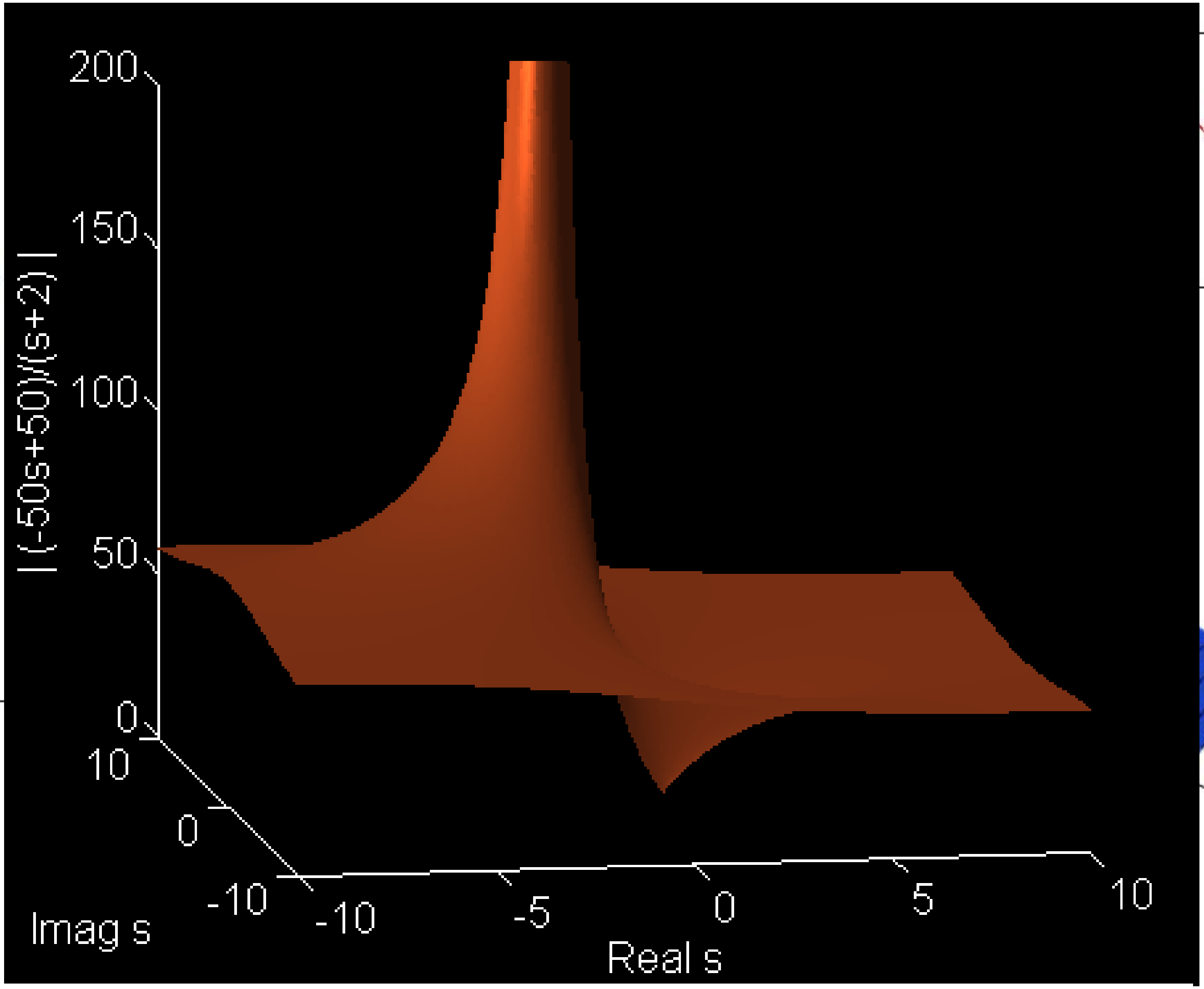
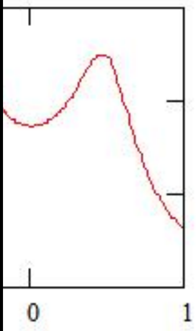
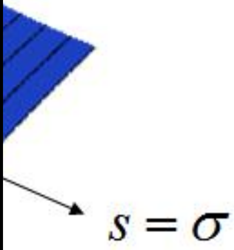
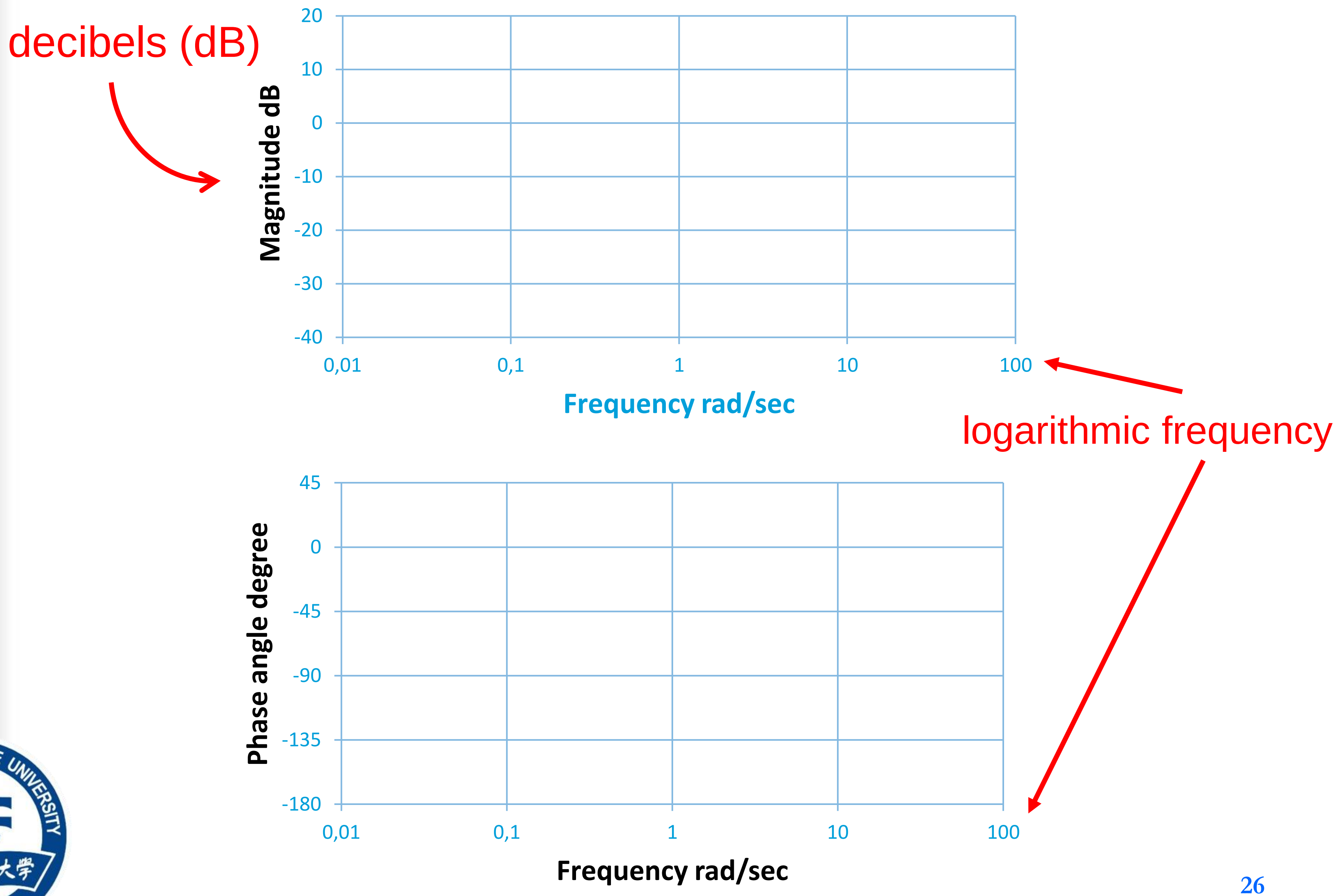






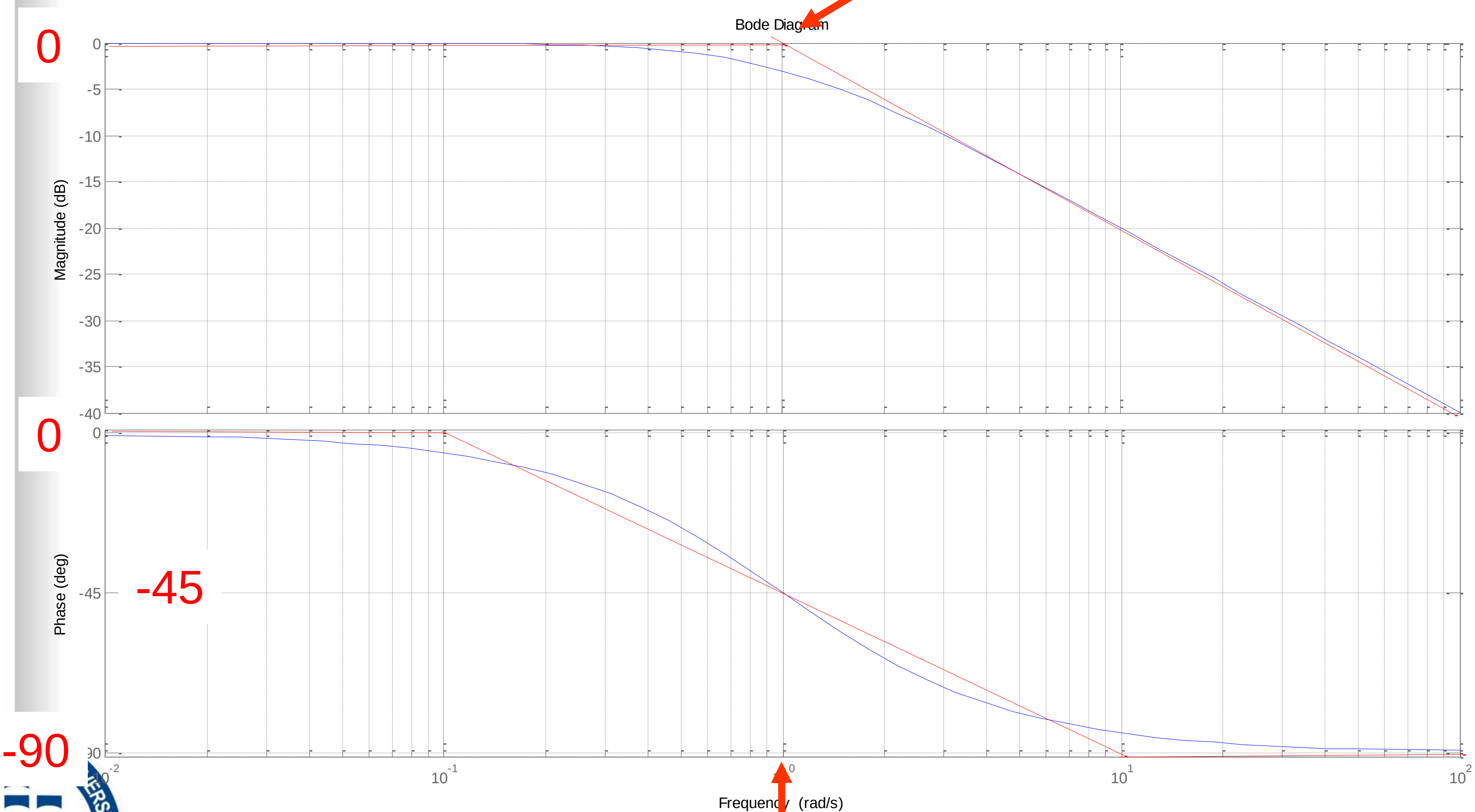
[source: Graphical Interpretation of Poles and Zeros]

Bode diagram



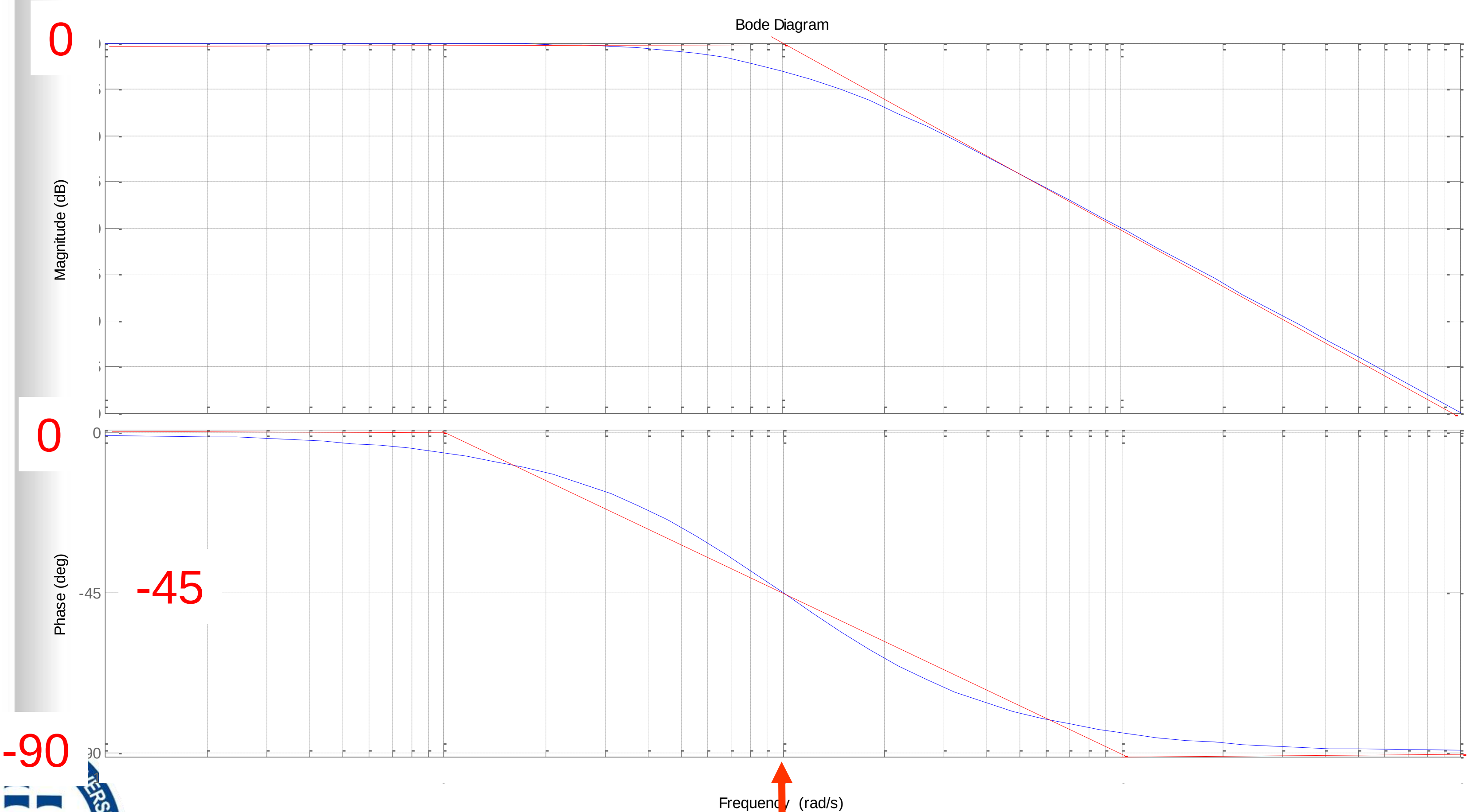
Bode diagram $\quad H(j\omega) = \dfrac{1}{j\omega + 1}$

change at corner frequency



$\omega = 1$

Bode diagram $\quad H(j\omega) = \dfrac{1}{\tau j\omega + 1}$

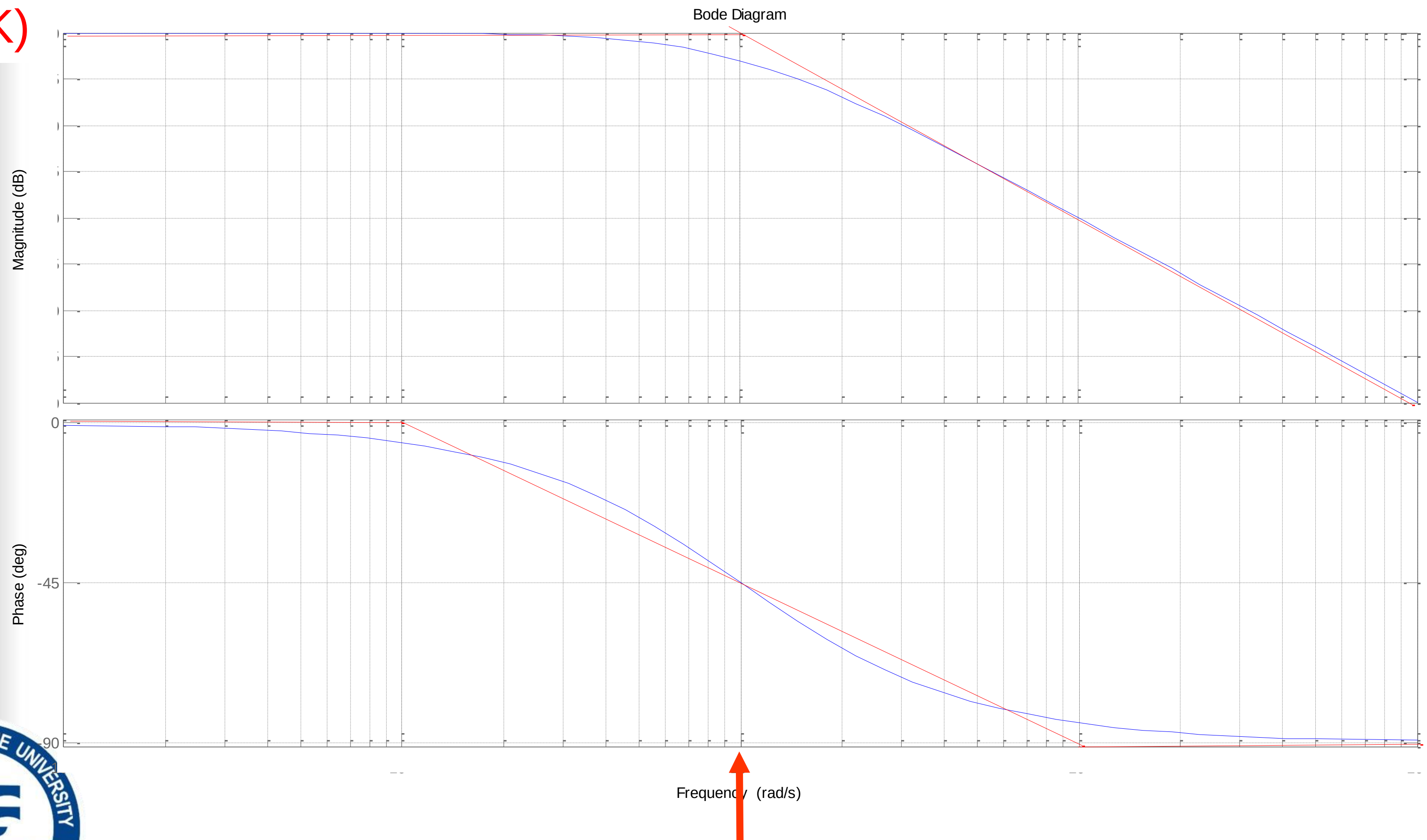


$\omega = 1/\tau$

Bode diagram

$$H(j\omega) = \frac{K}{\tau j\omega + 1}$$

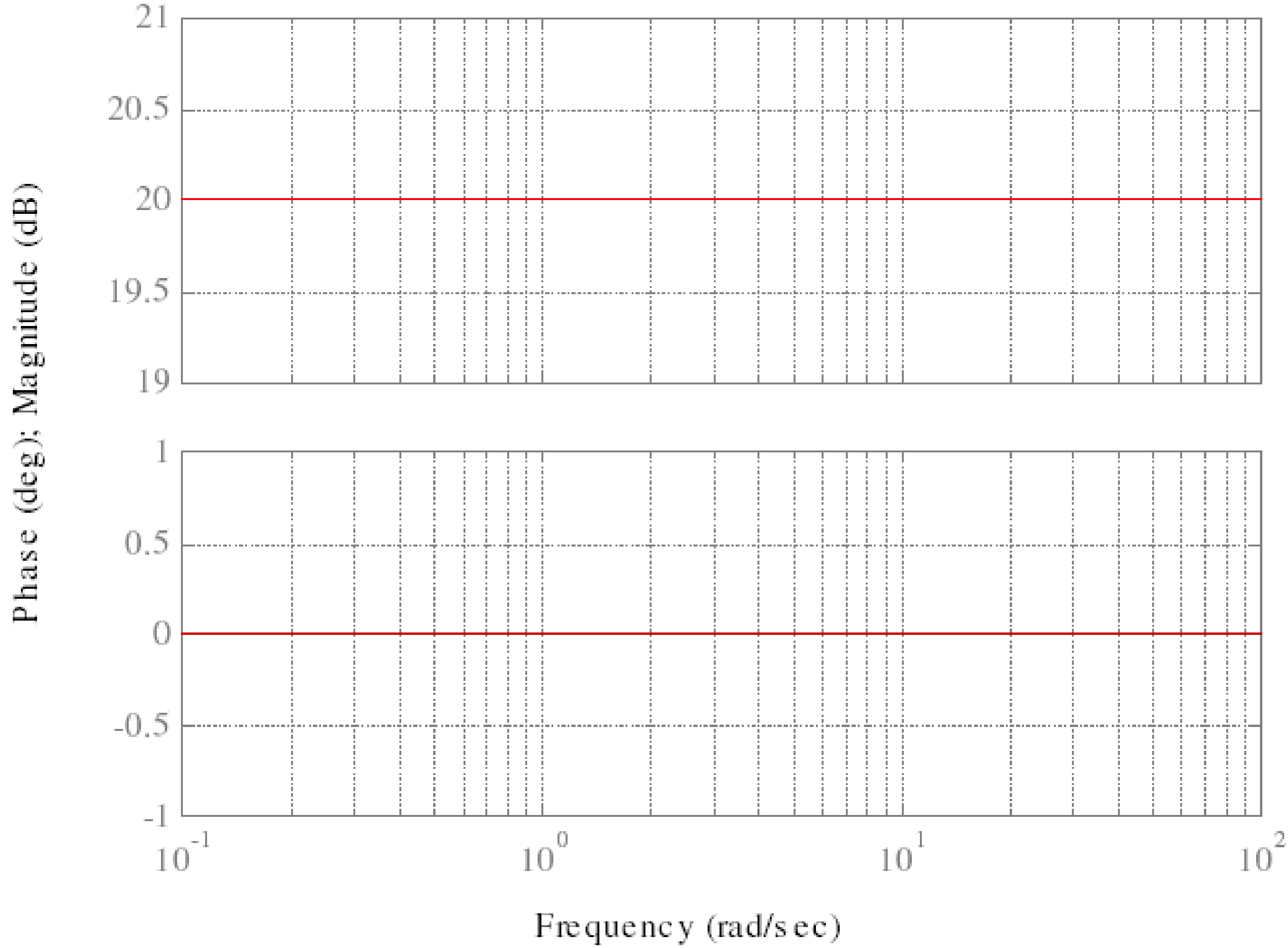
$20\log_{10}(K)$



$\omega = 1/\tau$

Bode diagrams examples

- Bode diagram for a constant gain; K = 10



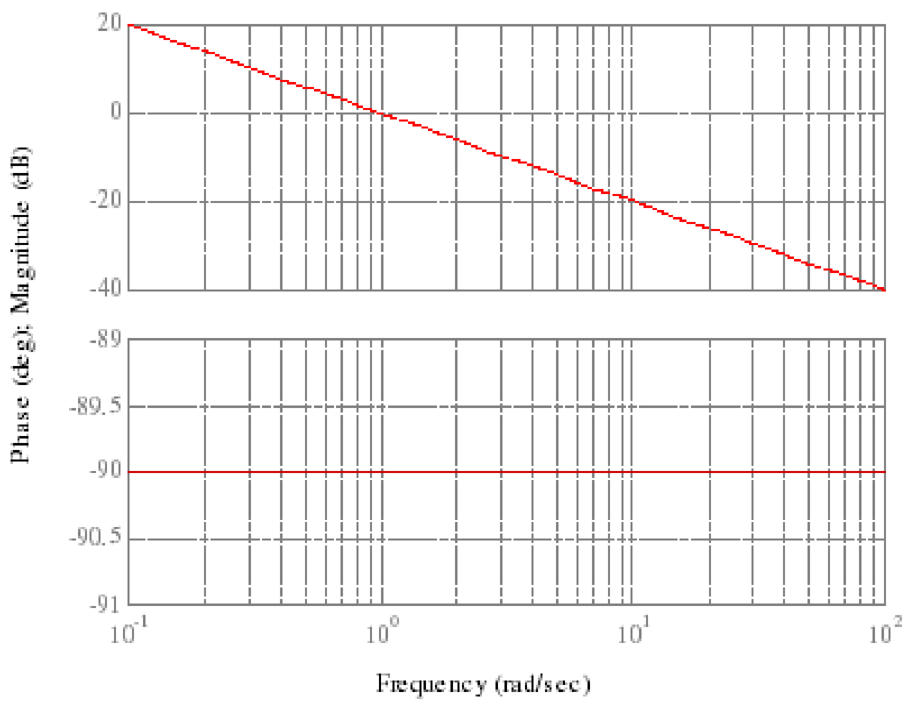
Bode diagrams examples

- Bode diagram for poles and zeros at the origin

  Slopes -20 dB/dec and +20 dB/dec

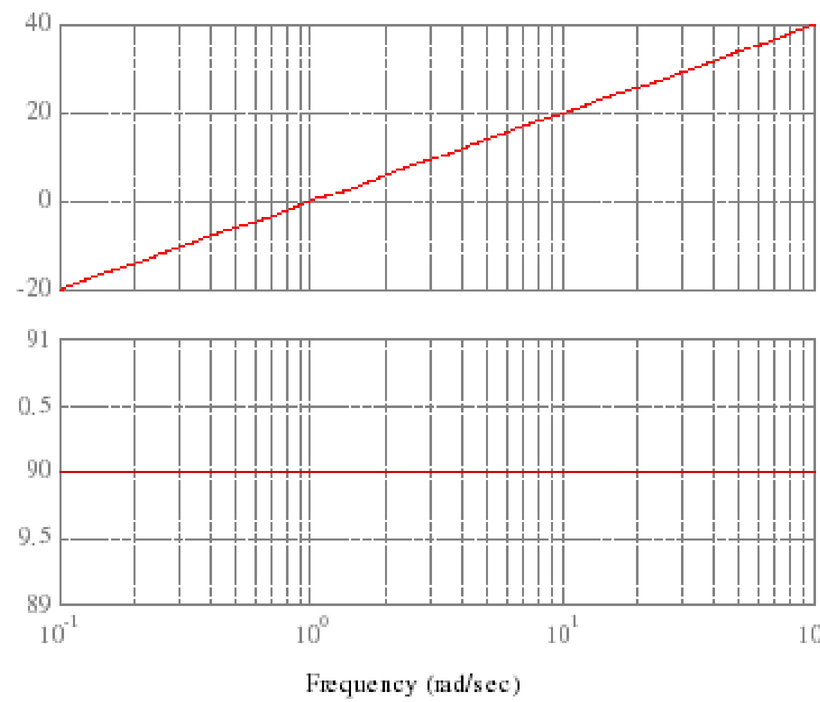
$$\boxed{G(s) = 1/s}$$

Integrator



$$\boxed{G(s) = s}$$

Differentiator



Bode diagrams examples

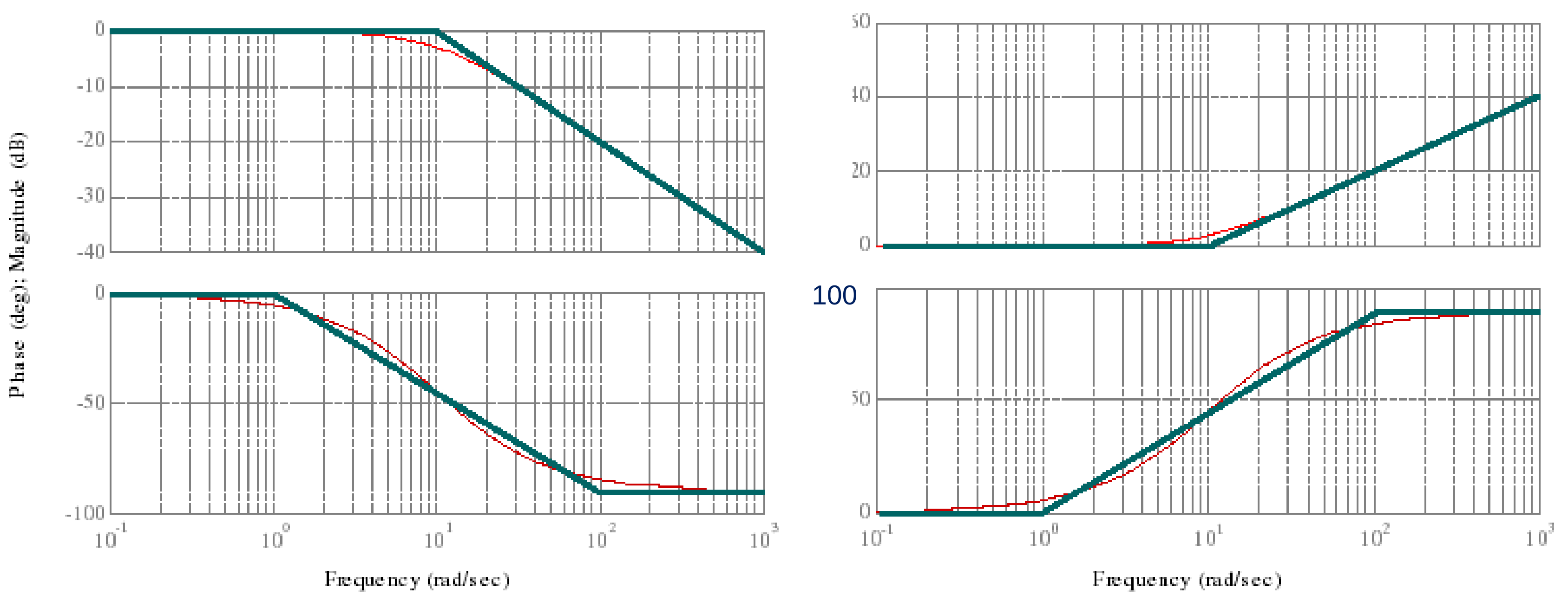
- Bode diagram for nonzero real poles and zeros
  - Questions:
    - What are the break (or corner) frequencies? 10 Hz
    - What are the slopes of the two magnitude plots? +/- 20dB/dec
    - What are the limits of the phase angles as $\omega \rightarrow \infty$? +/- 90 degrees

$$\boxed{G(s) = 10/(s+10)}$$

Low-pass filter

$$\boxed{G(s) = (s+10)/10}$$

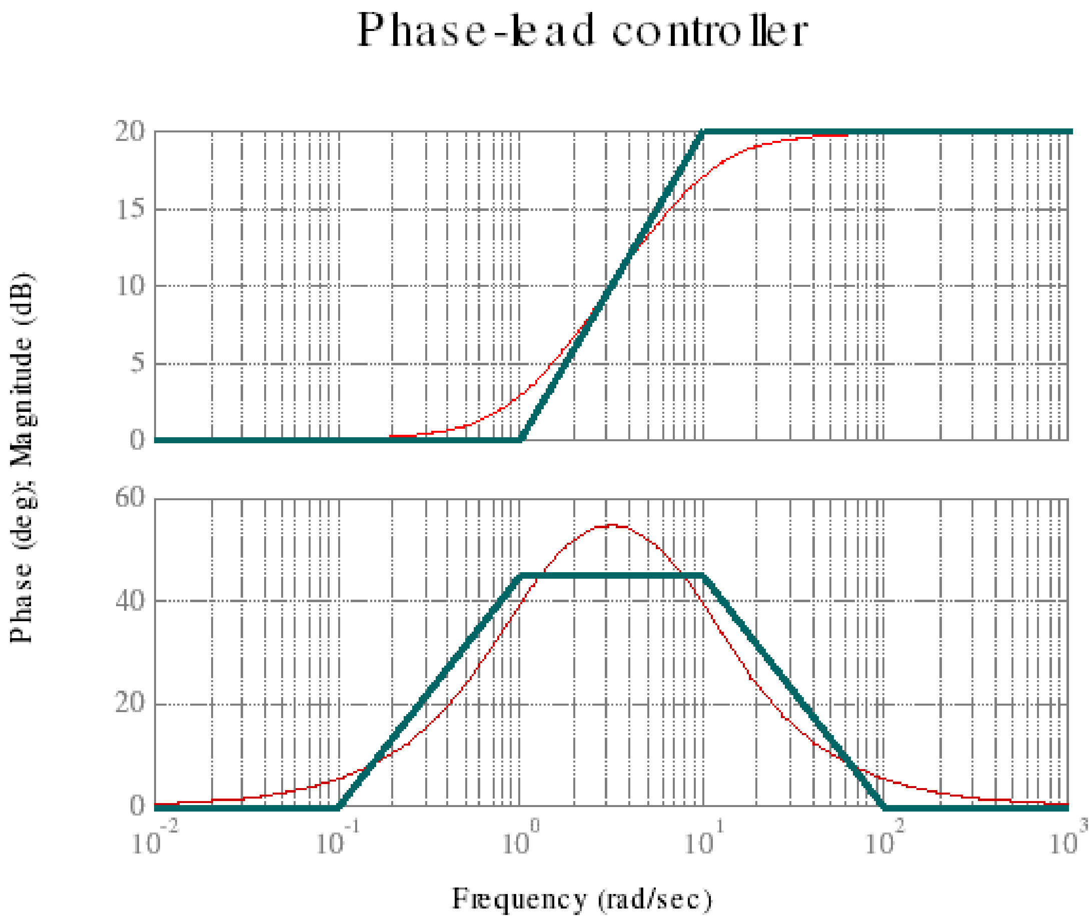
PD controller



Bode diagrams examples

- Bode diagram for nonzero real poles and zeros

$$G(s) = \frac{10(s+1)}{s+10}$$



Bode diagrams examples

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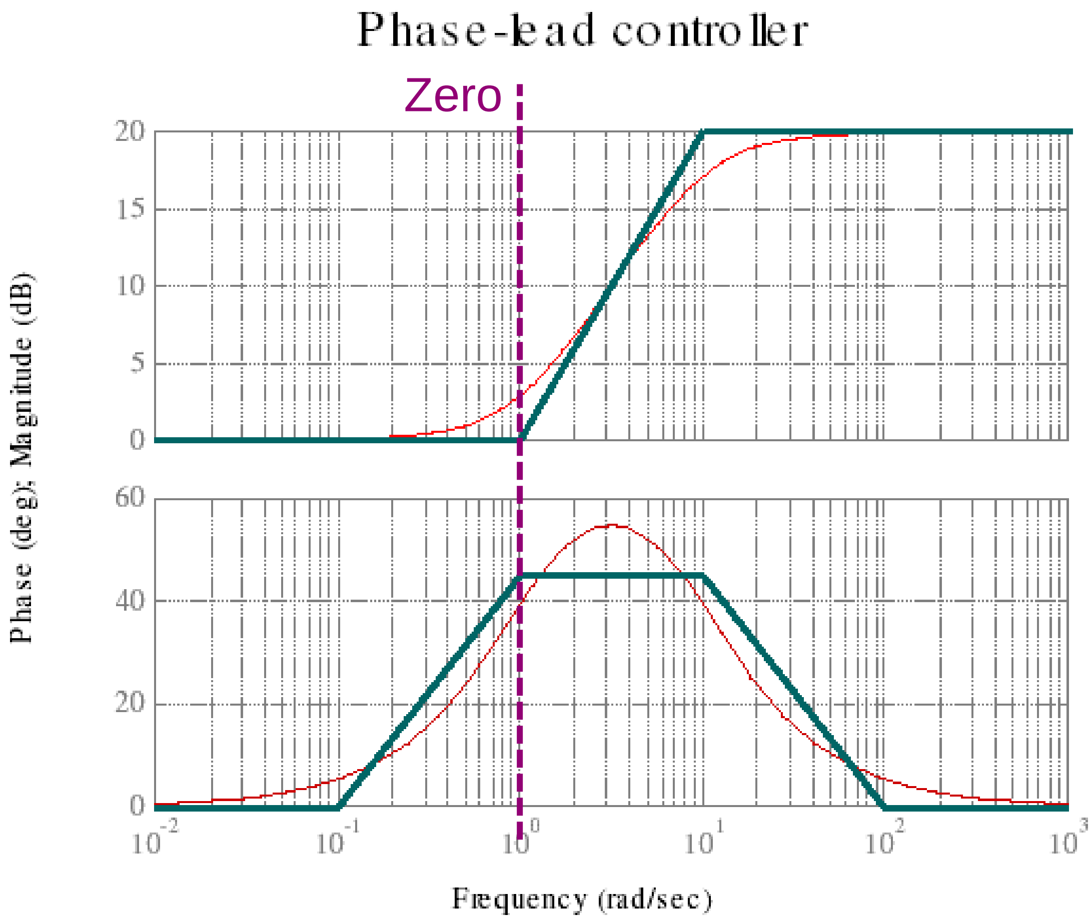

Bode diagrams examples

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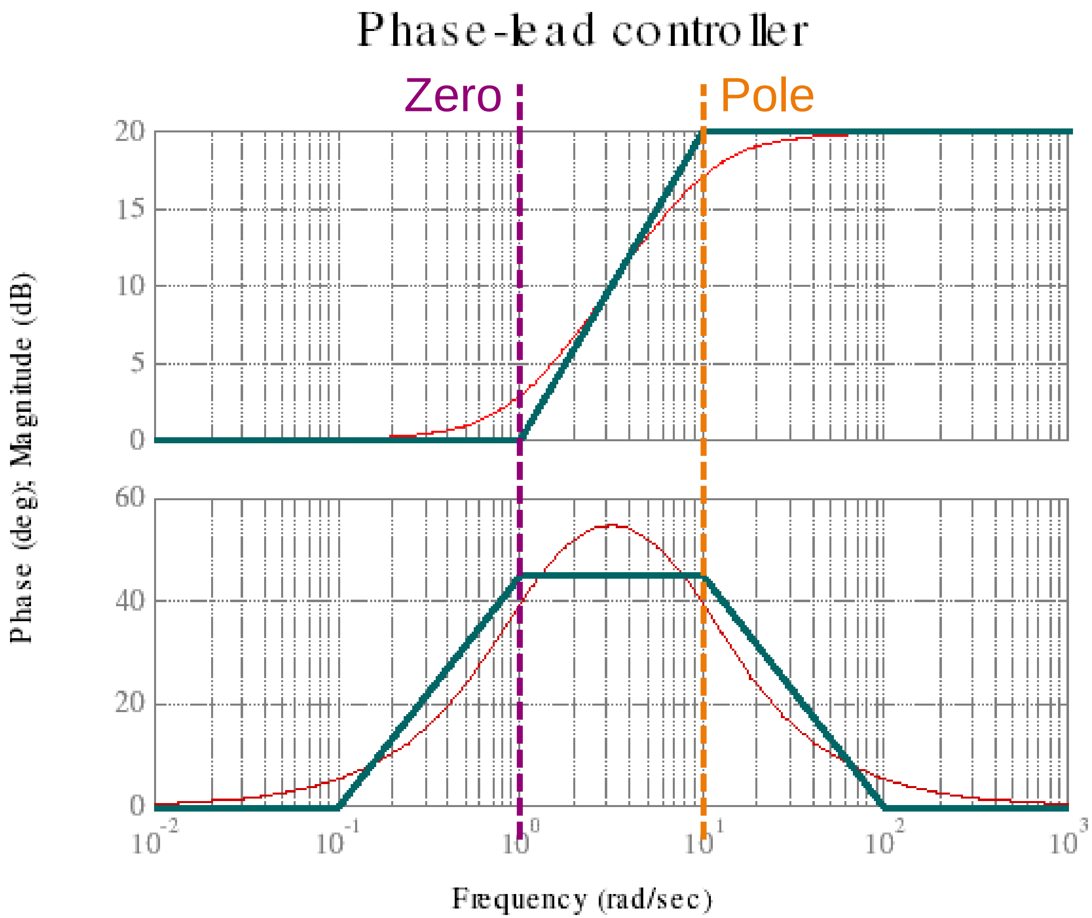
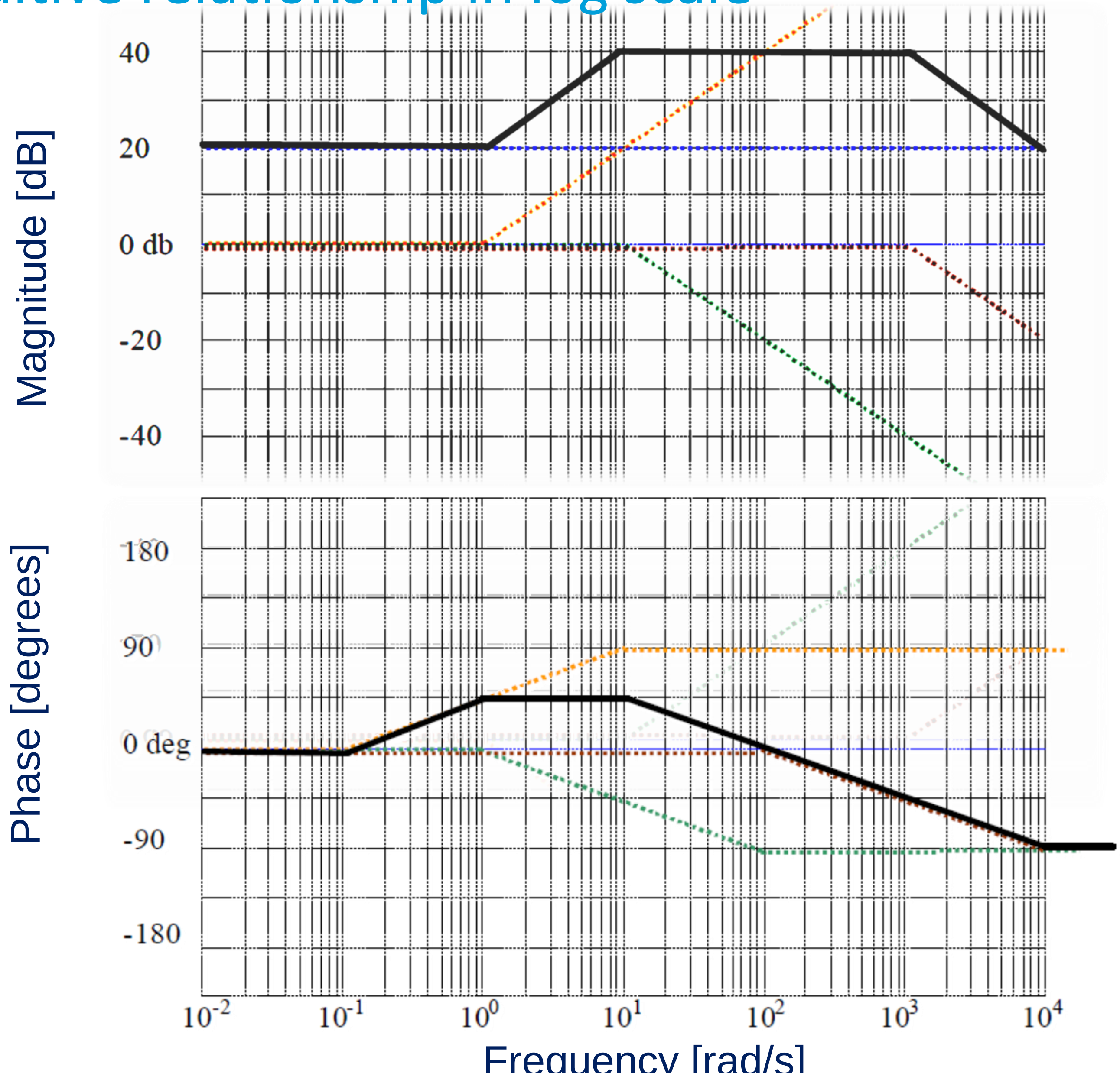
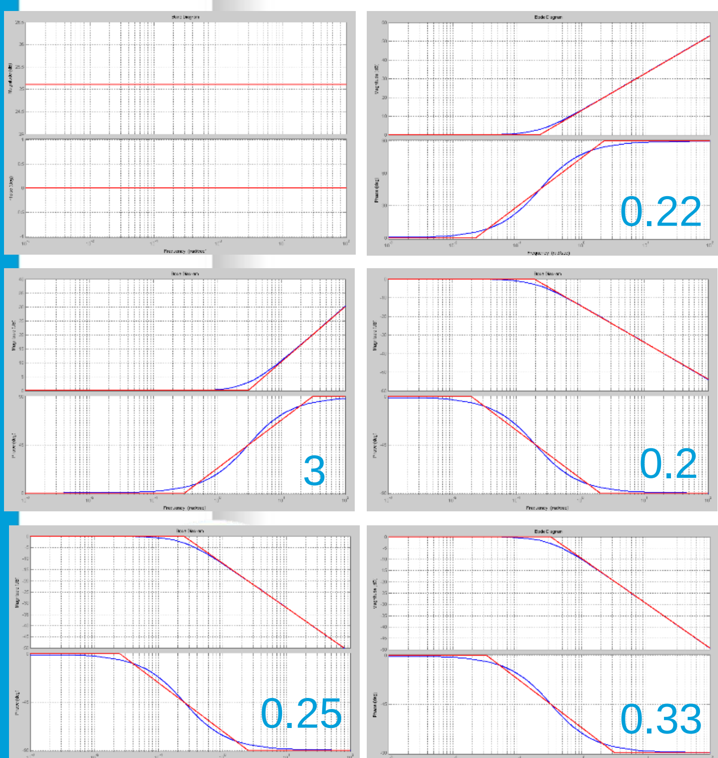


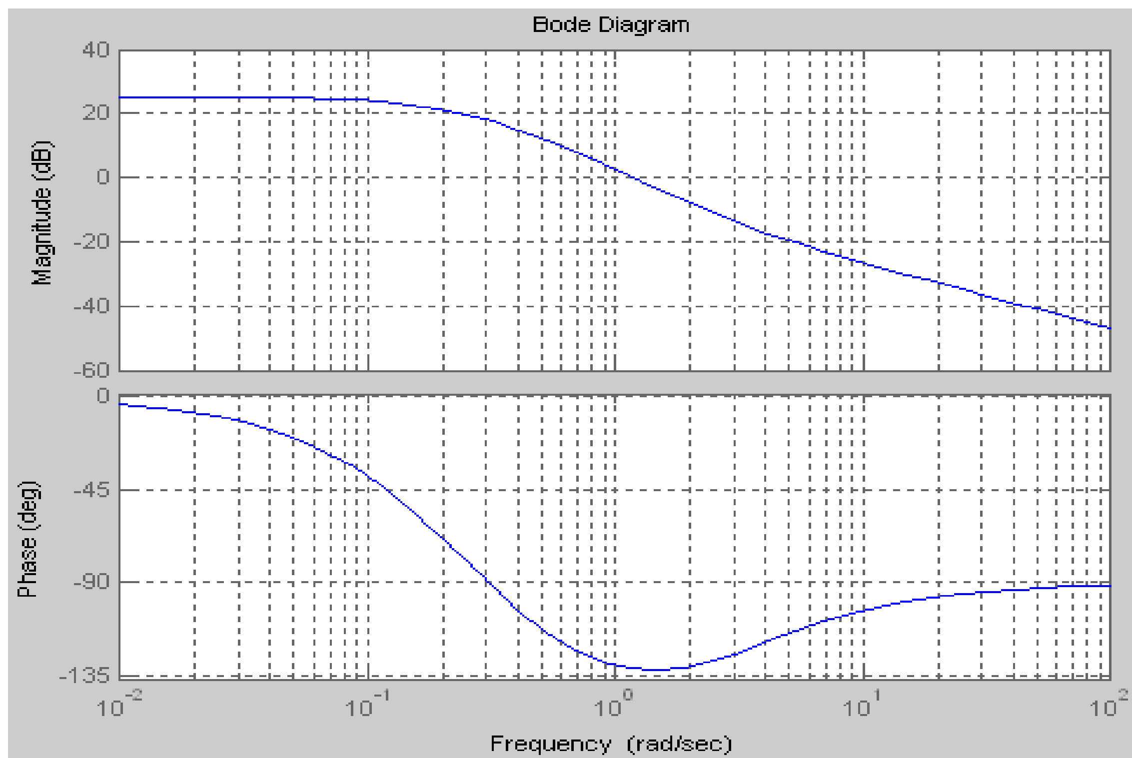
Bode plots example: additive relationship in log scale



Bode plots example: additive relationship in log scale

$$H(s)=\frac{27s^2+87s+18}{60s^3+47s^2+12s+1}=\frac{18(\frac{9}{2}s+1)(\frac{1}{3}s+1)}{(5s+1)(4s+1)(3s+1)}$$





Bode diagrams examples

Bode diagram for complex poles and zeros

- Consider poles or zeros of the form

$$s^2 + 2\beta\omega_0 s + \omega_0^2$$

Also written as: $s^2 + 2\zeta\omega_0 s + \omega_0^2$

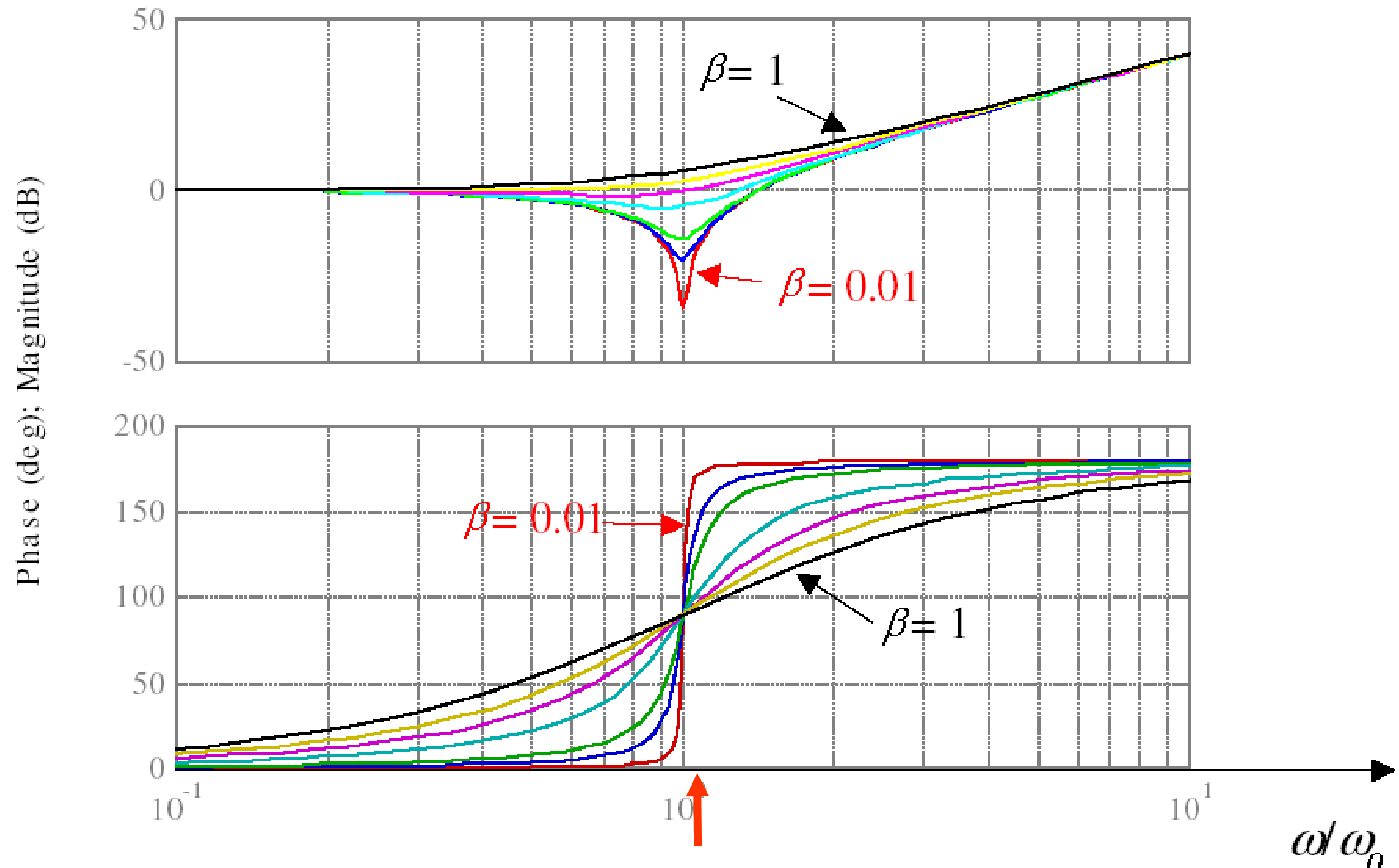
- For β<1 → Complex poles and zeros

- Straight-line approximations may be very inaccurate for some value of damping ration

Bode diagrams examples

Bode diagram for complex poles and zeros

$$G(s) = 1 + 2\beta \frac{s}{\omega_0} + \left(\frac{s}{\omega_0} \right)^2$$



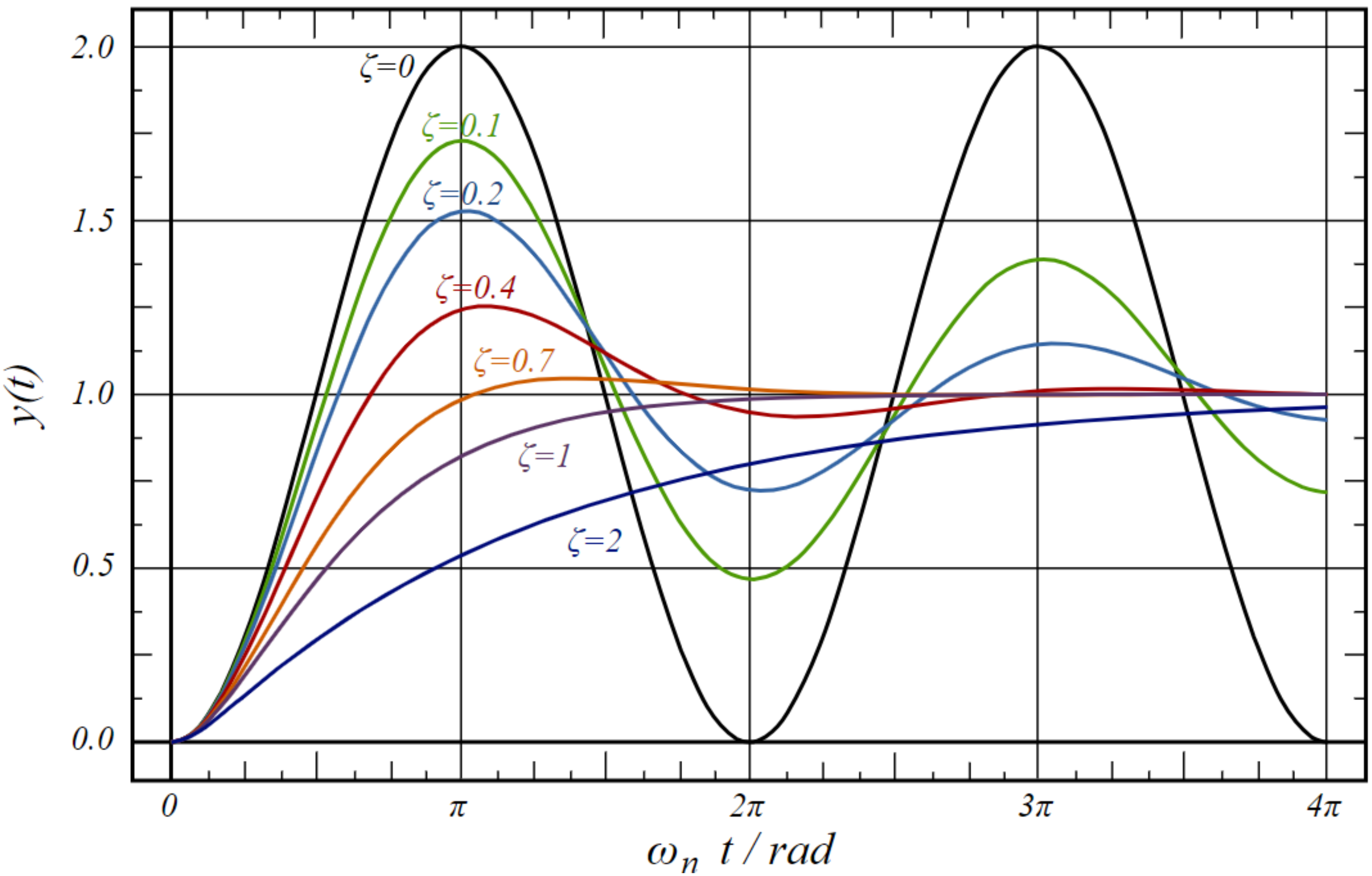
Peak is roughly at $\omega = \omega_0$

To be precise the lowest value for the magnitude is at $\omega = \omega_0 \mathrm{sqrt}(1-\beta^2)$, see Ogata p. 422

Bode diagrams examples

Effect on damping ration in the transient response of the system

$$G(s) = 1 + 2\zeta \frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2$$



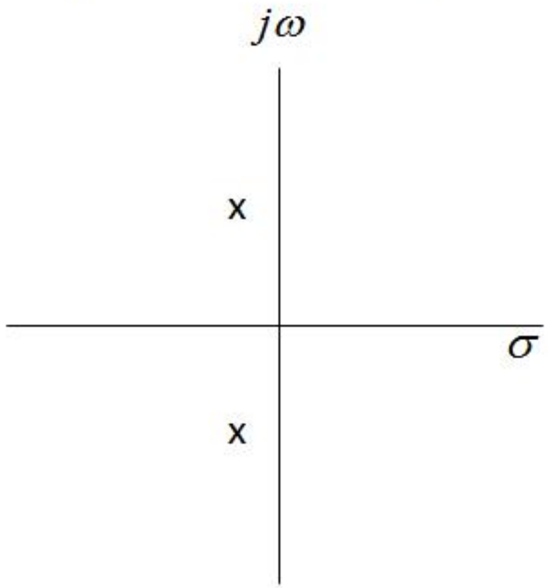
undamped ($\zeta = 0$), underdamped ($\zeta < 1$) through critically damped ($\zeta = 1$) to overdamped ($\zeta > 1$)

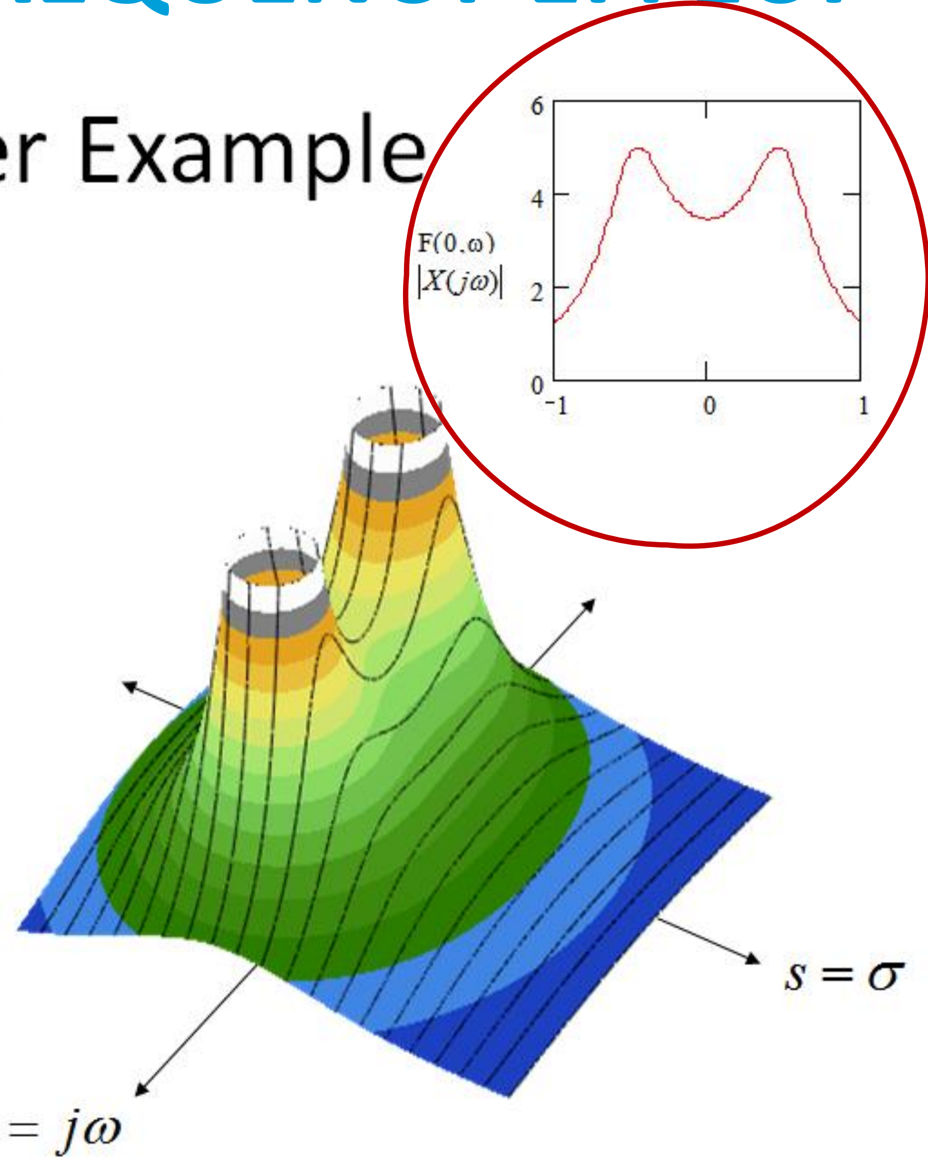
POLE ZERO FREQUENCY EFFECT

Filter Example

$$X(s) = \frac{1}{(s + .2 + j.5)(s + .2 - j.5)}$$

(poles at $s = -.2 \pm j.5$)





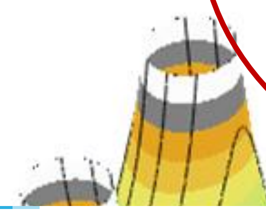
POLE ZERO FREQUENCY EFFECT

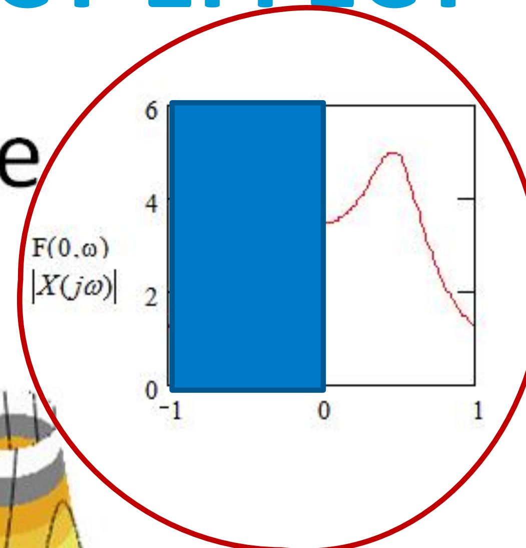
Filter Example

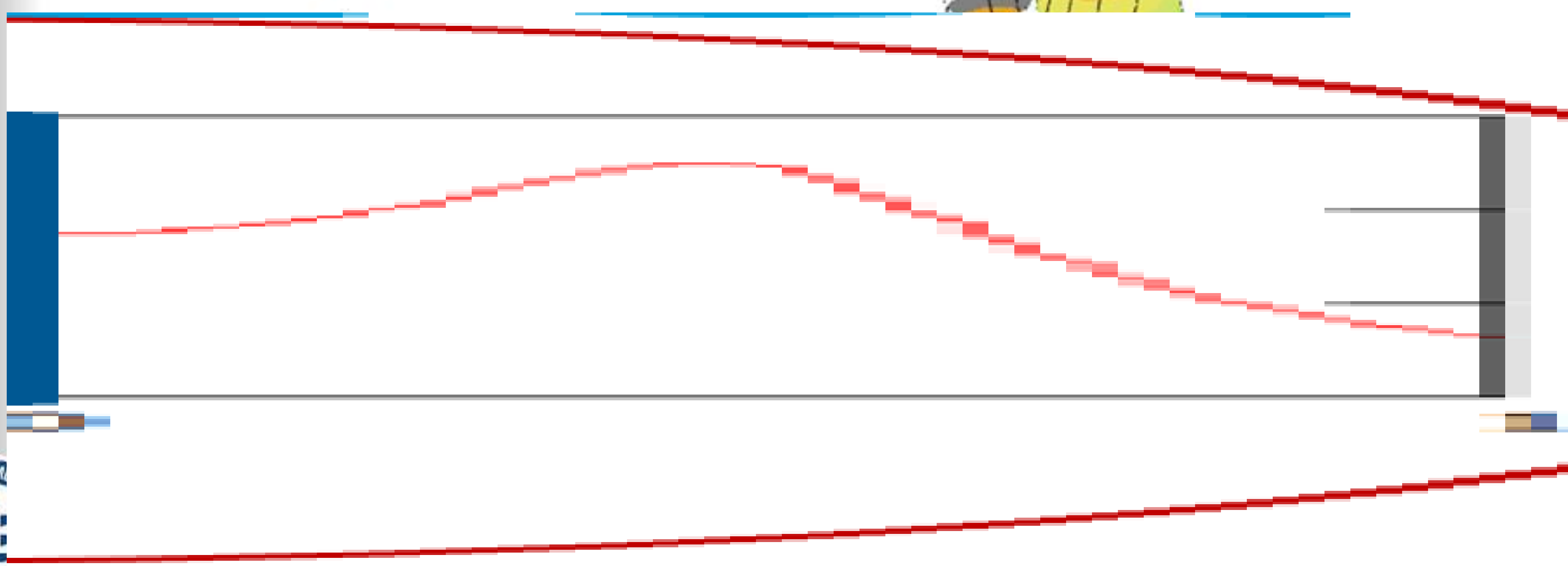
$$X(s) = \frac{1}{(s+.2+j.5)(s+.2-j.5)}$$

(poles at $s = -.2 \pm j.5$)

$j\omega$

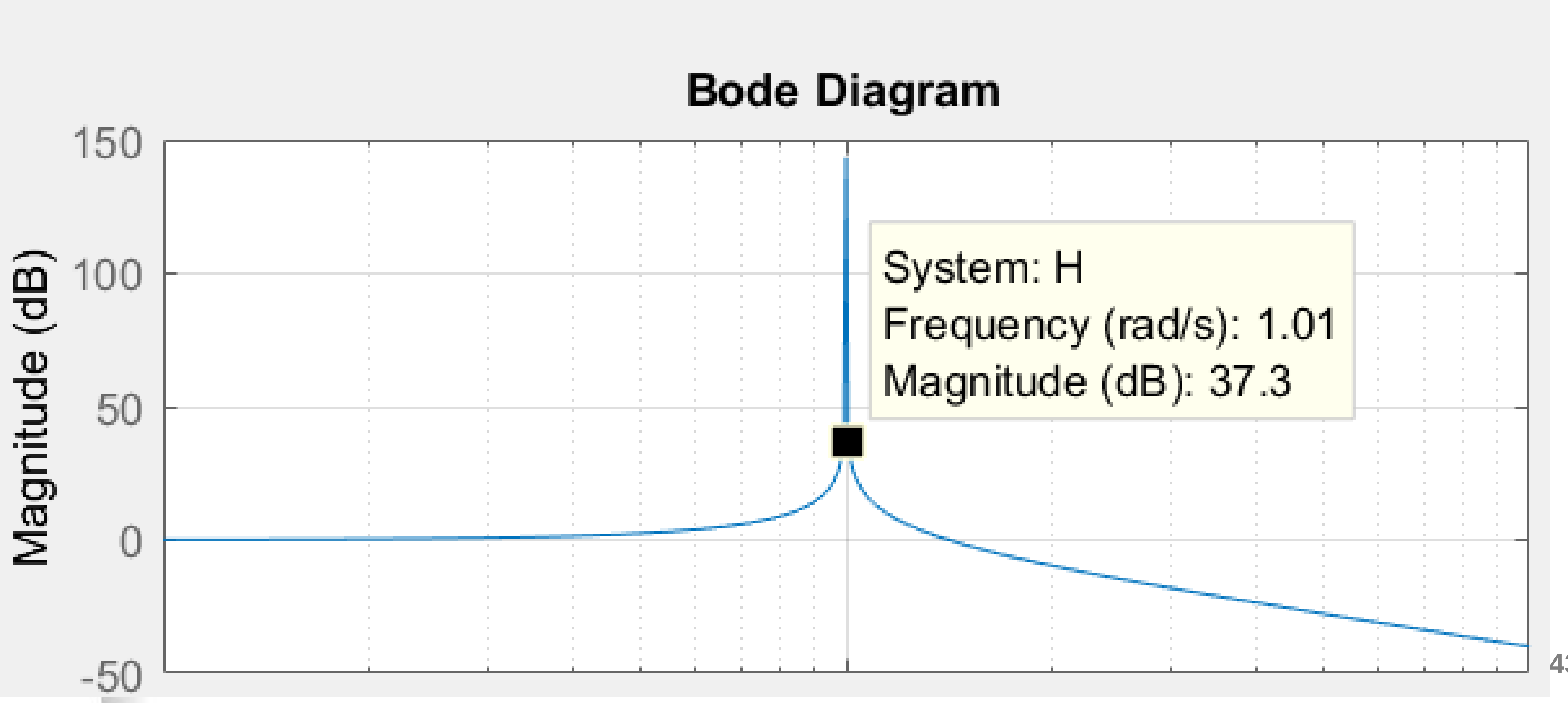






POLE ZERO FREQUENCY EFFECT

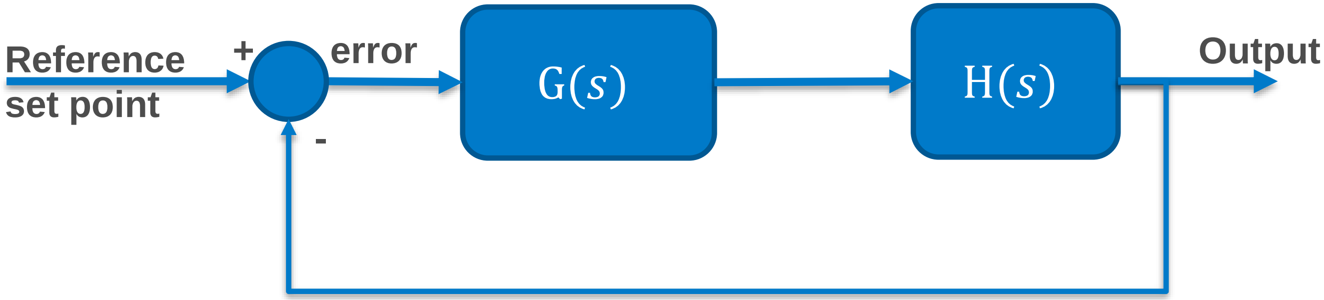
$$H(s) = \frac{1}{s^2 + 1}$$



CLOSED LOOP STABILITY

The closed loop transfer function:

$$CLTF = \frac{L(s)}{1 + L(s)}$$



We do **not** want:

$$CLTF \rightarrow \infty$$

$$\Rightarrow \quad L(s) \neq -1$$

$L(s) \neq -1$
WHAT DOES THIS MEAN ???

When:

$$L(s) = -1$$

We can infer:

1. $|L(s)| = 1$
2. $\angle L(s) = -180°$

Thus:

When magnitude is at $|L(s)| = 1$,

phase $\angle L(s)$ should not pass $-180°$

Before 180° phase, you should start suppressing your magnitude (signal).
Margin: How much more gain could you add to your system.

When phase $\angle L(s)$ is just at $-180°$,

magnitude $|L(s)|$ should not reach 1(0dB) or higher.

How much phase (time) do you have to suppress your signal till you reach 180° in phase.

$|L(s)| = 1, \angle L(s) = -180°$

When magnitude is at $|L(s)| = 1$,

phase $\angle L(s)$ should not pass $-180°$

Before $180°$ phase, you should start suppressing your magnitude (signal).
Margin: **How much** more gain could you add to your system until unstable.

Gain Margin(GM):

$$20\log_{10}\mathrm{GM} = 20\log_{10}1 - 20\log_{10}|L(s)\ \text{when}\ \angle L(s) = -180°|$$

When phase $\angle L(s)$ is just at $\pm 180°$,

magnitude $|L(s)|$ should not reach 1(0dB) or higher.

How much phase (time) do you have to suppress your signal till you reach -1(magnitude $= 1 = |-1|$) . So when magnitude reaches 1, you should already passed the $-180°$ phase.

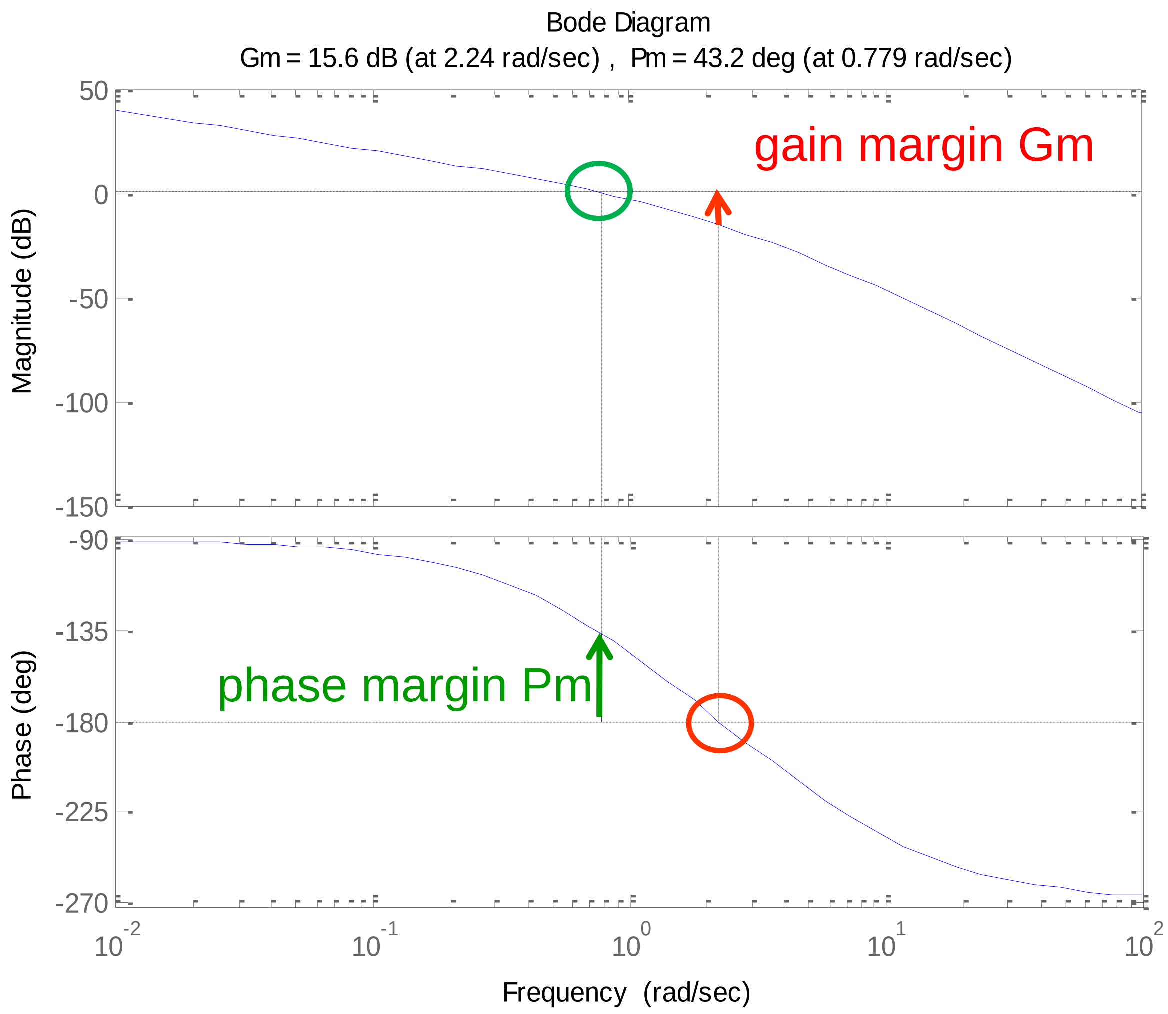
Phase Margin(PM):

$$\textbf{PM} = (-180°) - \angle(L(s)\ \text{when}\ |L(s)| = 1)$$

Stability margin and Matlab

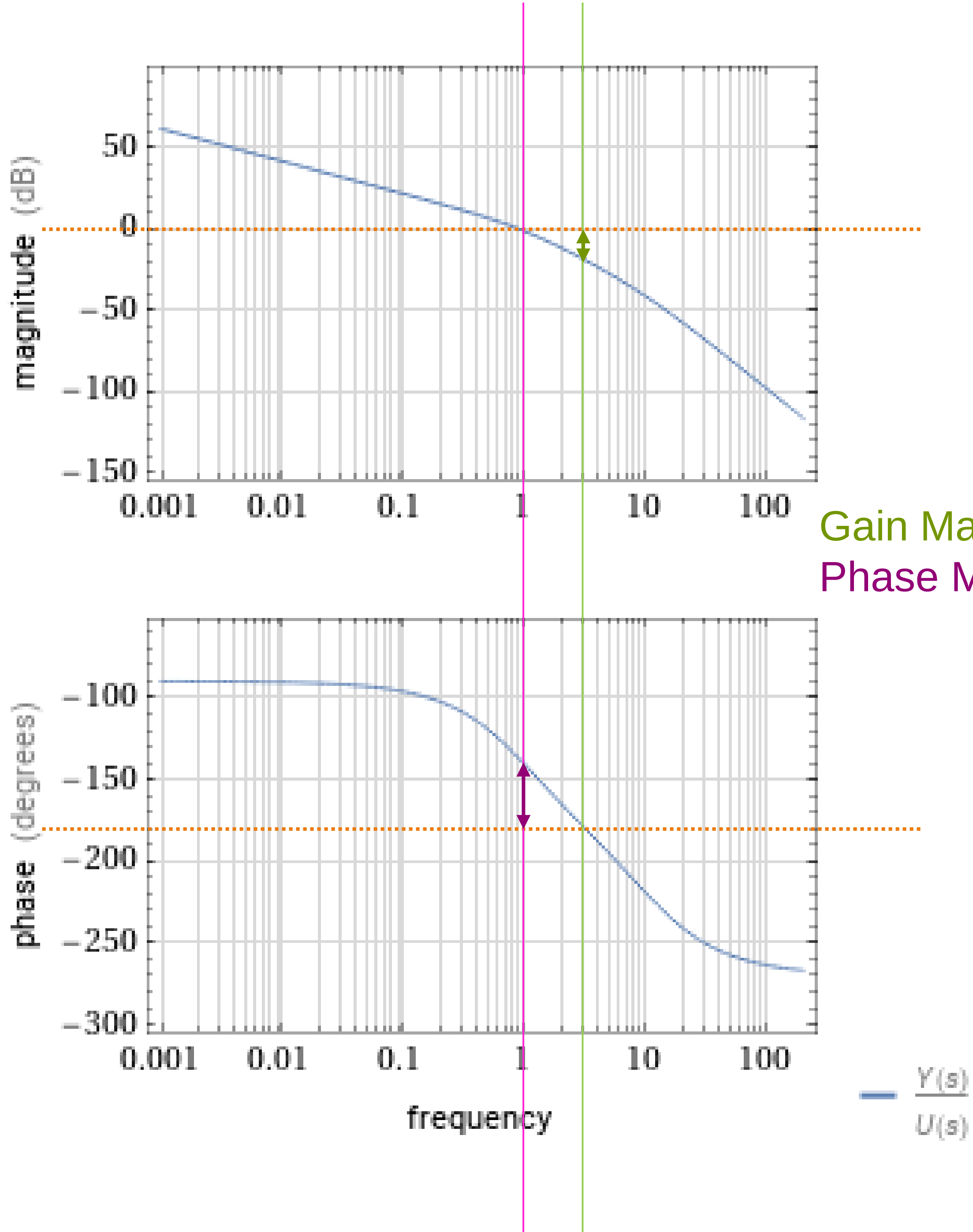
Matlab:

>> sys=tf(5,[1 6 5 0]), margin(sys)

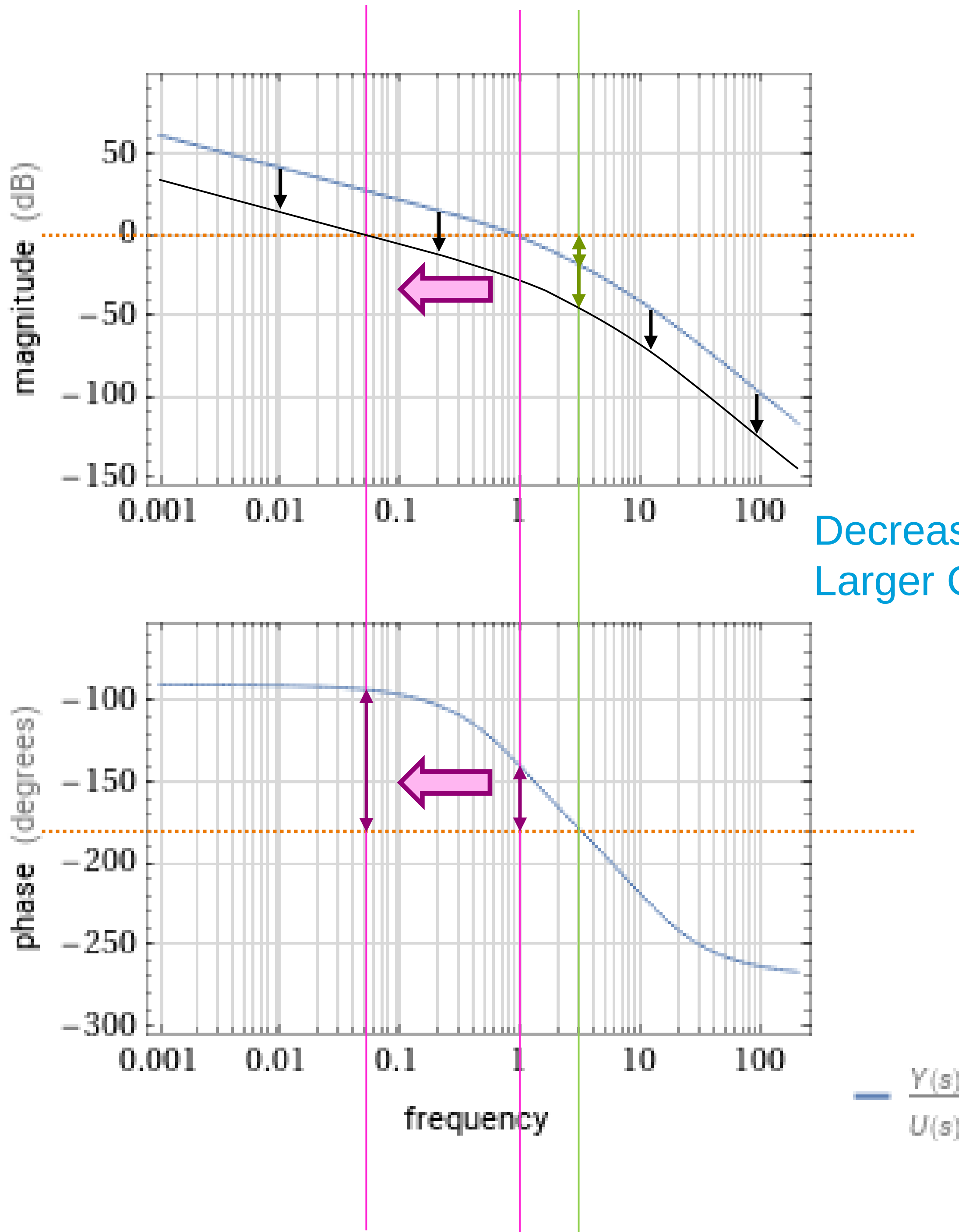


Minimum values requirement are often:
6 dB < Gm < 8 dB
45º < Pm < 65º

Of course, larger margins are safer.



Gain Margin 19.2 dB
Phase Margin 43 deg



Decreased K >>
Larger GM and PM

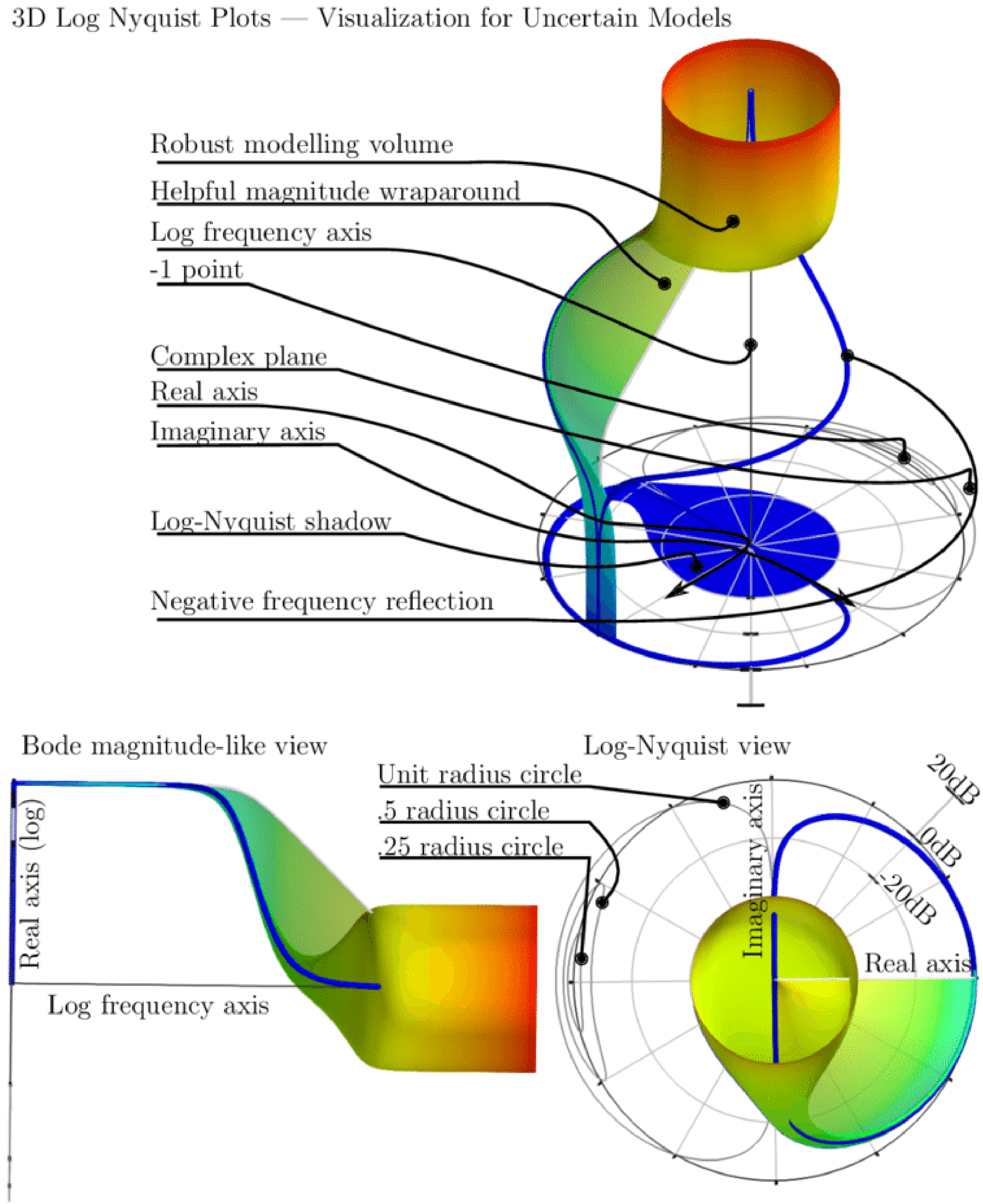
$L(j\omega)$ is so important

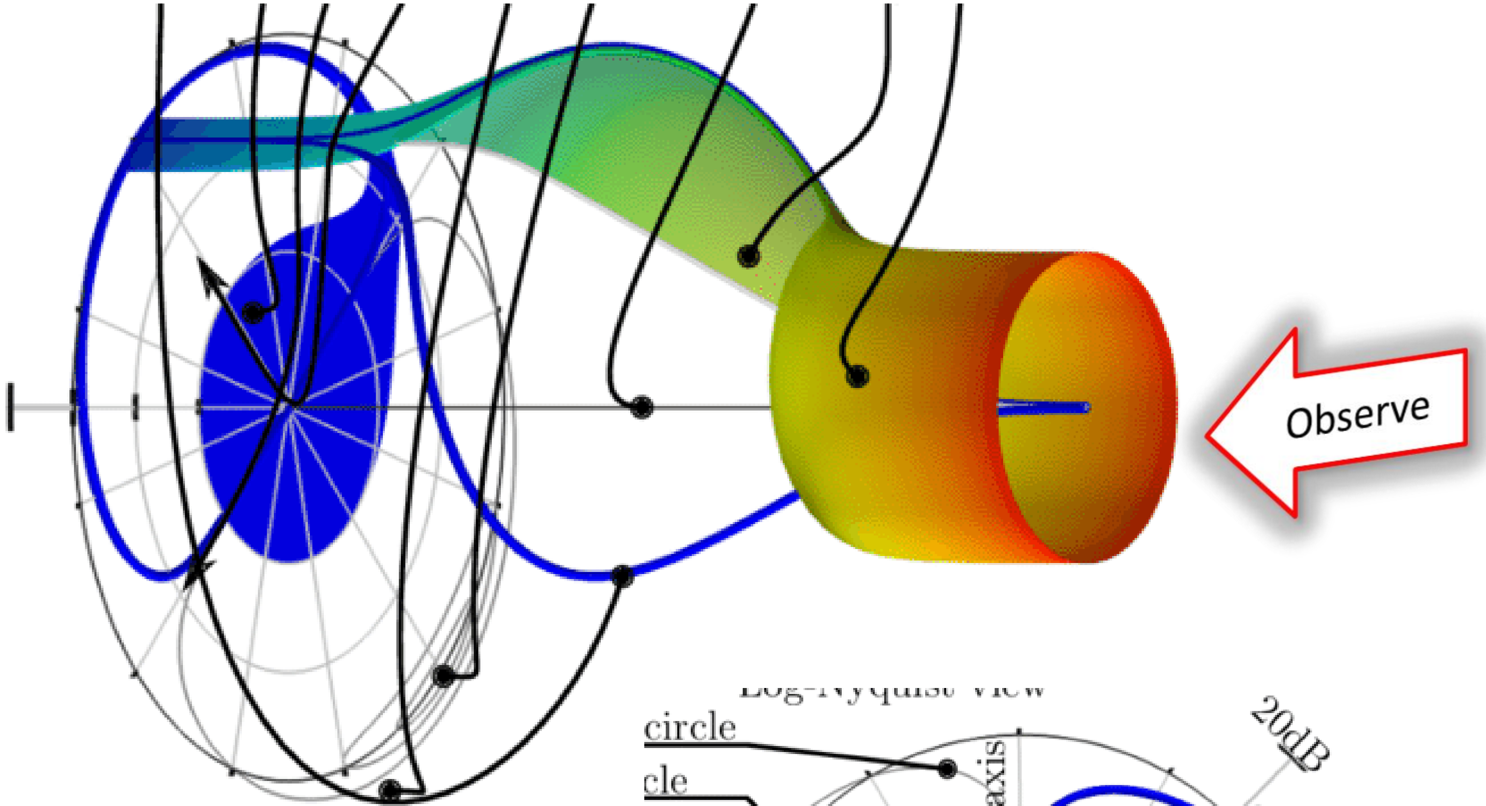
Why don't we plot it?

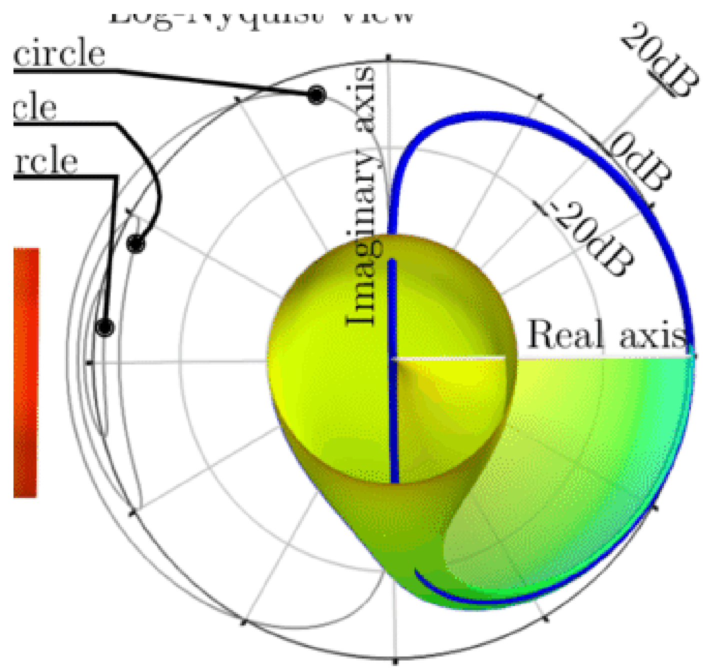
$L(j\omega)$ is a complex function about ω

3D plot

Source:
Control Strategies for Series Elastic, Multi-Contact Robots,
Gary Thomas, 2019,
doctorale dissertation,
University of Texas Austin







We care about amplitude and angle of the transfer function.

So for convenience we project the 3D plot to 2D plot in the complex plane for $L(j\omega)$

Nyquist plot

A Nyquist plot shows on the complex plane the real part of a frequency response function against its imaginary part with frequency as an implicit variable.

The Nyquist plot works with the **open loop** transfer

$$KG(s)H(s) = \frac{K}{s+1}$$

Substitute $s = j\omega$ To look at the forced response

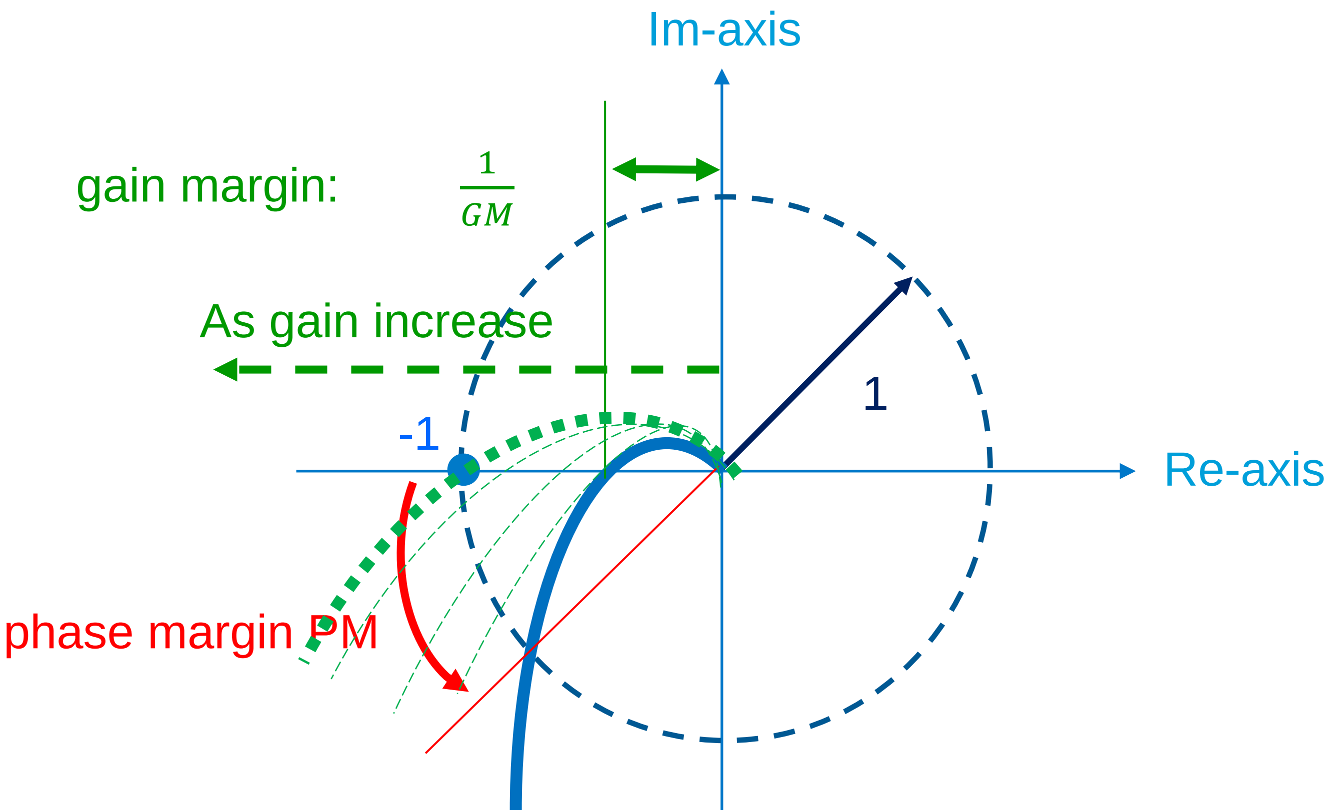
Multiply with the complex conjugate to separate real and imaginary parts

$$KG(j\omega)H(j\omega) = \frac{K}{\omega^2+1}(1-j\omega) = \frac{K}{\omega^2+1} - j\frac{K\omega}{\omega^2+1}$$

Stability margins in the Nyquist plot

GM: gain margin is the distance to $|KG(j\omega)H(j\omega)| = 1$ for a phase of -180º

PM: phase margin is the distance to a phase of -180º for $|KG(j\omega)H(j\omega)| = 1$

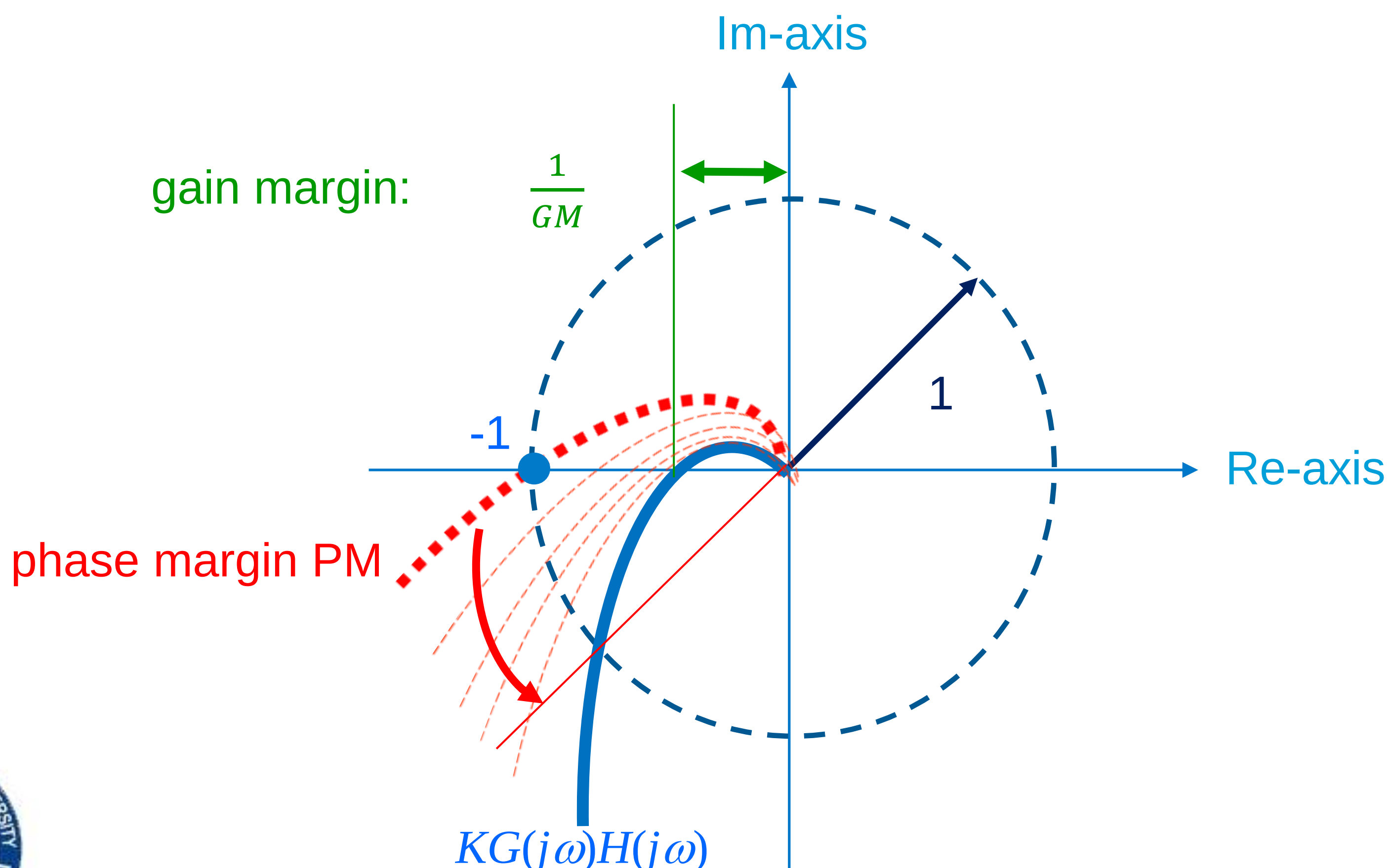


$L(s) = KG(j\omega)H(j\omega)$

Stability margins in the Nyquist plot

GM: gain margin is the distance to $|KG(j\omega)H(j\omega)| = 1$ for a phase of -180º

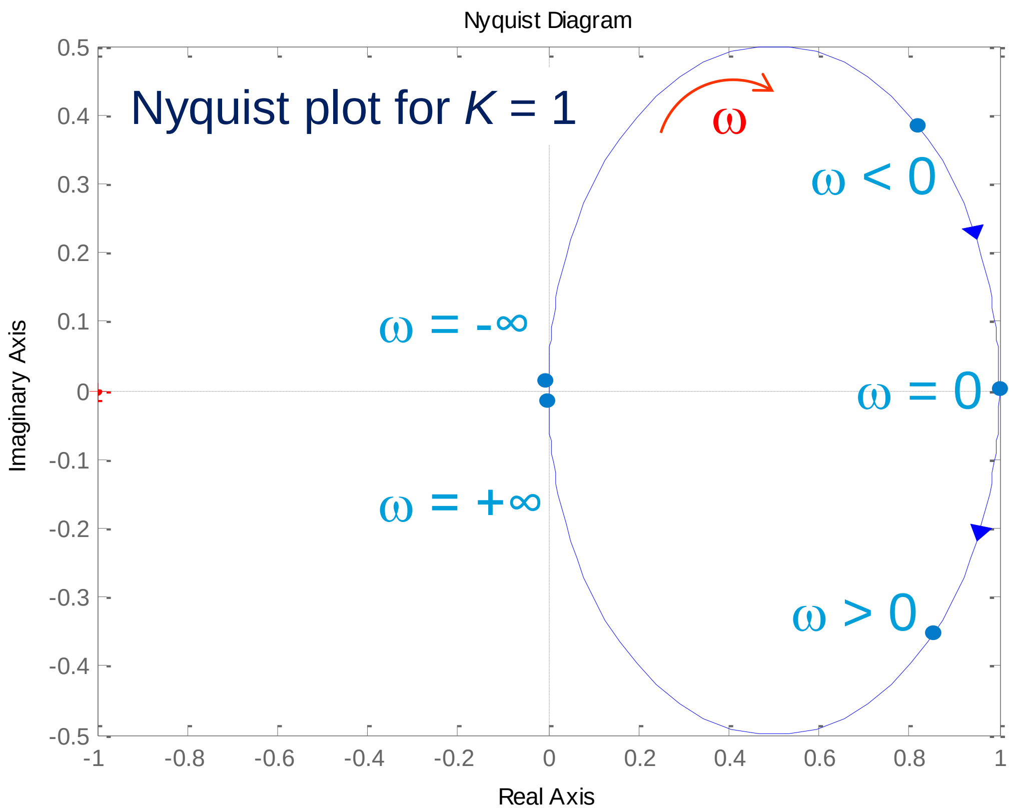
PM: phase margin is the distance to a phase of -180º for $|KG(j\omega)H(j\omega)| = 1$



Nyquist plot

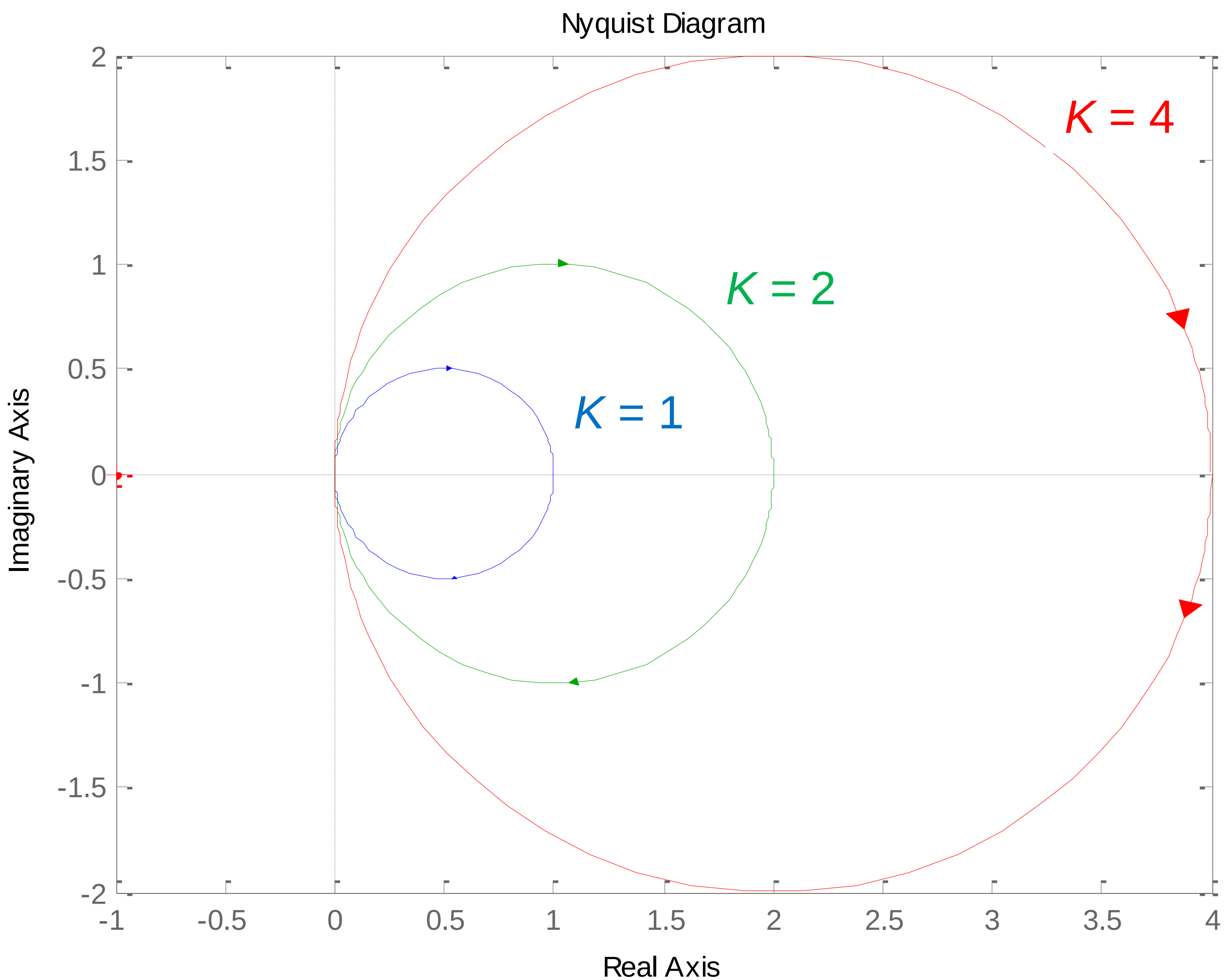
The changes of the complex value of *KG*(*jw*)*H*(*jw*) gives a shape in the complex plane, and this shape is the Nyquist plot.

$$KG(j\omega)H(j\omega)=\frac{K}{\omega^2+1}(1-j\omega)=\frac{K}{\omega^2+1}-j\frac{K\omega}{\omega^2+1}$$



Nyquist plot

The shape of the Nyquist plot changes with different parameter settings of the controller



Nyquist plot

Our open loop transfer function is now written in a real and an imaginary part

$$KG(j\omega)H(j\omega) = \frac{K}{\omega^2 + 1} - j\frac{K\omega}{\omega^2 + 1}$$

plotting our open loop transfer function on the complex plane while increasing ω from -∞ to +∞ will result in the Nyquist plot.

$$\left.\begin{aligned} &\text{For } \omega = -\infty \rightarrow \ KG(-\omega)H(-\omega) = 0 - j0 \\ &\text{For } \omega = +\infty \rightarrow \ KG(+\omega)H(+\omega) = 0 - j0 \end{aligned}\right\}$$

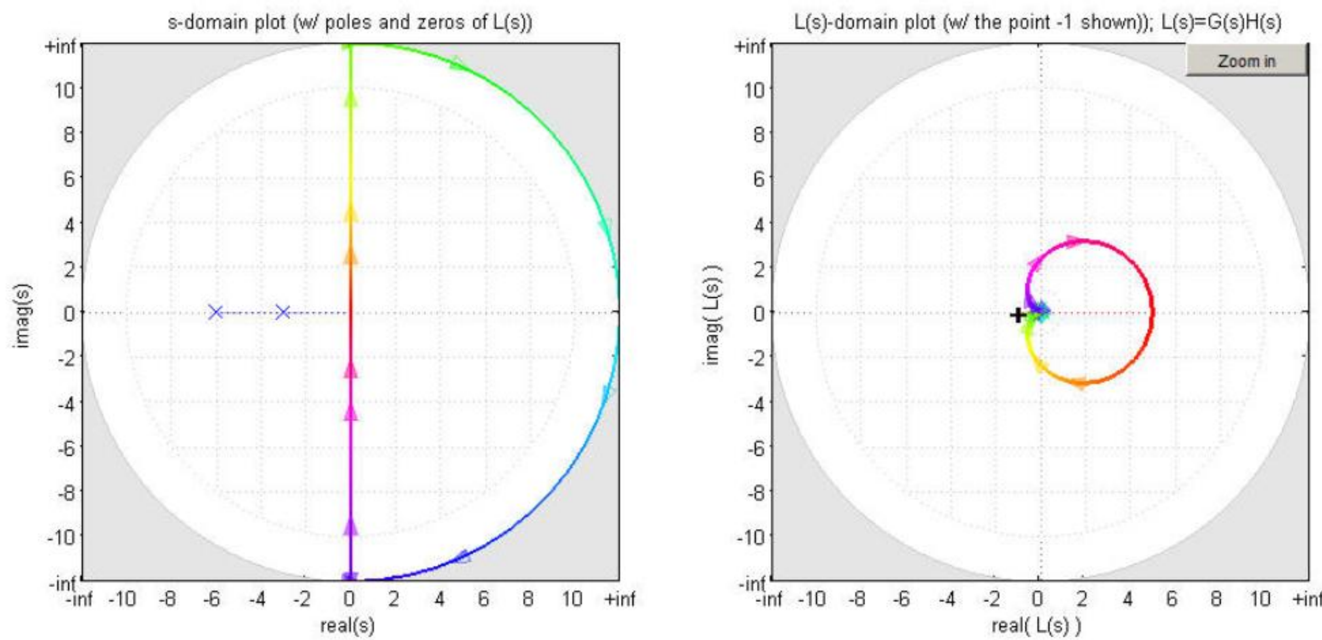
$-\infty$ and $+\infty$ Are at the same point closing the contour

For $\omega = 0 \quad \rightarrow \ KG(0)H(0) = K - j0$

Cauchy's principle of argument

To find out if our system is stable we are going to look for poles and zeros in the right half plane RHP.

$$L(s) = \frac{90}{s^2 + 9s + 18} = \frac{90}{(s+3)(s+6)}$$

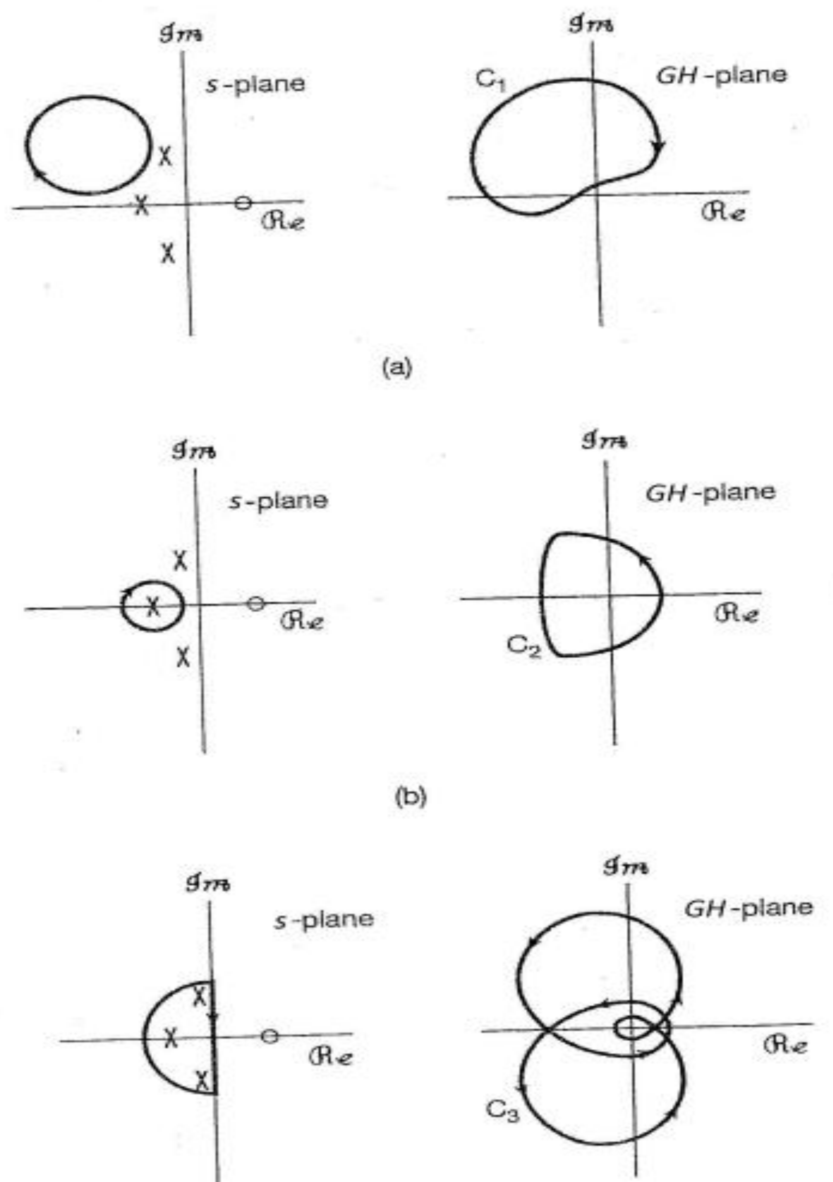


A contour map of a complex function, for example the function KG(jw)H(jw), will encircle the origin
[Z – P] times, where Z is the number of zeros and P the number of poles of the function inside the contour.

Cauchy's principle of argument: Mapping by F(s)

Mapping:

$$GH(s) = \frac{s-1}{(s+1)(s^2+s+1)}$$

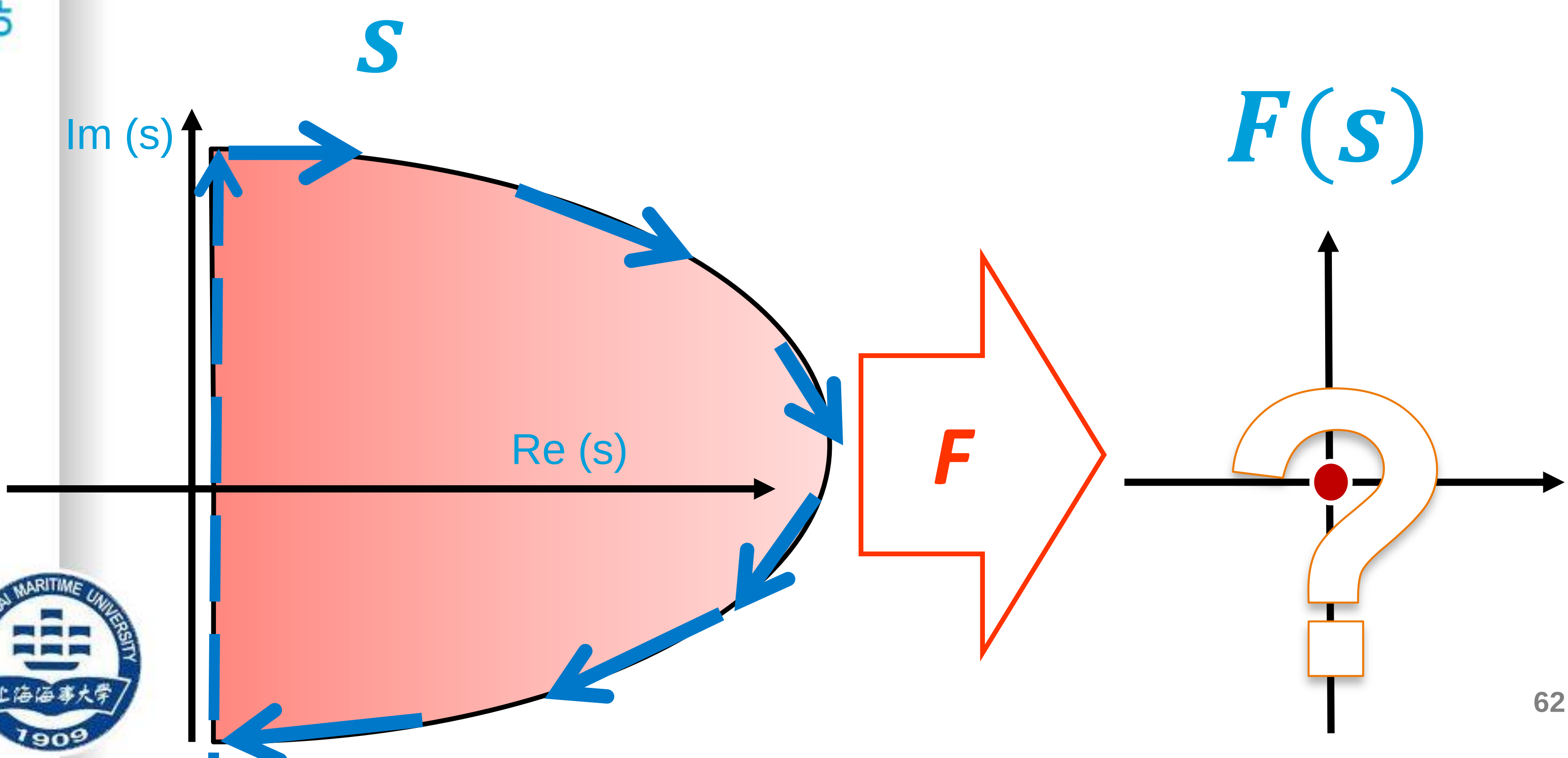


A. V. Oppenheim, A. S. Willsky with S. H. Nawab, Signals & Systems, 2nd ed., Prentice Hall, 1997, page 849

Cauchy's principle of argument: Mapping by F(s)

To find out if our system is stable we are going to look for poles and zeros in the right half plane RHP.

Mapping:



CLOSED LOOP STABILITY

Open Loop transfer function

$$L(s) = \frac{N(s)}{D(s)}$$

Closed Loop transfer function

$$\frac{L(s)}{1 + L(s)} = \frac{N(s)}{D(s) + N(s)}$$

CLOSED LOOP TF POLES AND ZEROS

We were working with open loop transfer function

$$OLTF = G(s)H(s) = K\frac{N(s)}{D(s)}$$

The closed loop transfer function:

$$CLTF = \frac{G(s)H(s)}{1+G(s)H(s)} = \frac{K\frac{N(s)}{D(s)}}{1+K\frac{N(s)}{D(s)}} = \frac{KN(s)}{D(s)+KN(s)}$$

POLES MOVE!
ZEROS STAYS!

After this slide we looked at **Root Locus**

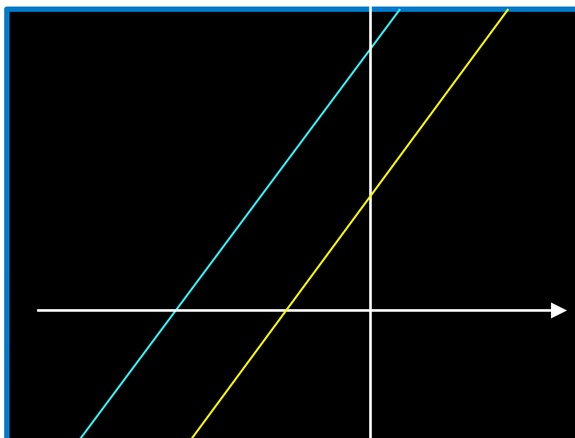
CLOSED LOOP STABILITY

OLTF: $L(s) = K\frac{N(s)}{D(s)}$, CLTF: $\frac{L(s)}{1+L(s)} = \frac{KN(s)}{\boxed{D(s)+KN(s)}}$

Observation: $1 + L(s)$

$$1 + L(s) = \frac{\boxed{D(s) + KN(s)}}{D(s)}$$

Zeros	of	L(s)	are the closed loop zeros
Poles	of	L(s)	are the 1+L(s) poles
Zeros	of	1+L(s)	are the closed loop poles



$$y = x + 1$$
$$(0,1), (-1,0), (-2,-1)$$
$$\gamma = y + 1 = x + 2$$
$$(0,2), (-1,1), (-2,0)$$

γ is just y shifted 1 unit to the left

CLOSED LOOP STABILITY

Zeros of	L(s)	are the closed loop	zeros
Poles of	L(s)	are the 1+L(s)	poles
Zeros of	1+L(s)	are the closed loop	poles

Looking at properties of **1+L(s) in Nyquist plot of L(s):**

Number of CW encirclement of L(s) at -1 =
{number of zeros of 1+L(s) – number of poles of 1+L(s)}.

Closed loop stability requirement:

no CLTF poles in RHP

CLOSED LOOP STABILITY

Zeros	of	L(s)	are the closed loop zeros
Poles	of	L(s)	are the 1+L(s) poles
Zeros	of	1+L(s)	are the closed loop poles

Looking at properties of **1+L(s) in Nyquist plot of L(s)**:

Number of CW encirclement of Nyquist plot of L(s) at -1

= {number of zeros of 1+L(s) in RHP

– number of poles of 1+L(s) in RHP}.

Closed loop stability requirement:

no CLTF poles in RHP

CLOSED LOOP STABILITY

Zeros of	L(s)	are the closed loop	zeros
Poles of	L(s)	are the 1+L(s)	poles
Zeros of	1+L(s)	are the closed loop	poles

Looking at properties of **1+L(s) in Nyquist plot of L(s)**:

Number of CW encirclement of Nyquist plot of L(s) at -1

= {number of zeros of 1+L(s) in RHP

– number of poles of 1+L(s) in RHP}.

Closed loop stability requirement:

no CLTF poles in RHP

Zeros of 1+L(s) – **P**oles of L(s) =

No. CW encirclement at -1

CLOSED LOOP STABILITY

Zeros of 1+L(s) – **P**oles of L(s) = **N**o. CW encirclement at -1

We don't want poles in the RHP.

Zeros of 1+L(s) =

(CLTF Poles)

Poles of L(s) + **N**o. CW encirclement at -1

(OLTF Poles) (Nyquist plot characteristic)

Nyquist stability criterion

Z = P + N

Closed loop stable iff Z=0

CLOSED LOOP STABILITY

Zeros of 1+L(s) – **P**oles of L(s) = **N**o. CW encirclement at -1

We don't want poles in the RHP.

Zeros of 1+L(s) =

(CLTF Poles)

Poles of L(s) + **N**o. CW encirclement at -1

(OLTF Poles) (Nyquist plot characteristic)

Nyquist stability criterion

Z = P + N

Closed loop stable iff Z=0

Equivalent to N = -P

Nyquist stability criterium

Observe the Nyquist plot

$$Z_{RHP} = N_{CWE} + P_{OL\_RHP}$$

the closed-loop system is unstable if Z > 0

$Z_{RHP} = Number\ of\ closed\ loop\ poles\ in\ the\ Right\ Half\ Plane$

$N_{CWE} = Number\ of\ Clock\ Wise\ Encirclements\ of\ the\ point\ -1+j0$

$P_{OL\_RHP} = Number\ of\ poles\ of\ the\ Open\ Loop\ system\ in\ the\ Right\ Half\ Plane$

If encirclements are in the counterclockwise direction, N_{CWE} is negative

The P_{OL_RHP} is not shown in the Nyquist plot but is found from then transfer function

Why encircle the point -1+j0?

$$\Delta(s) = 1 + KG(s)H(s) = 0 \longrightarrow KG(s)H(s) = -1$$

NYQUIST & BODE VS ROOT LOCUS

From root locus there is another famous stability test that is convenient called Routh-Hurwitz stability criterion.

We can not deal with time delay in Root Locus.
Root Locus only deal with rational functions with **polynomials on both numerators and denominators**.

We do have an approximation method, called Pade's approximation. By Taylor's series expansion, we may approximate e^{-sT} to the form:

$$e^{-sT} \approx K\frac{s+p}{s+q} = -1\frac{s-\frac{2}{T}}{s+\frac{2}{T}}$$

NYQUIST & BODE
VS
ROOT LOCUS

Nyquist & bode plots, work with all $\boldsymbol{L(s)}$.

We only need magnitude and phase,

$$|\boldsymbol{L(s)}| \quad , \quad \angle \boldsymbol{L(s)}$$

And the open loop

With $s = j\omega$ and experimental measurement,

sometimes without explicitly knowing the transfer function,

we may infer the stability of the system!

(The open loop poles you can read from Bode plots)

NYQUIST & BODE VS ROOT LOCUS

When to use what?

Root Locus: Design

When you have a open loop transfer function and would like to design a system and determine the adequate controller gain K.

Bode & Nyquist: Evaluation

You already have the controller and gain parameter K or just an overall unknown open loop system, you would like to see if the closed loop system is stable or not. And evaluate the robustness of your system: gain margin & stability margin.