BASIC CONTROL SYSTEMS

07 ROOT LOCUS

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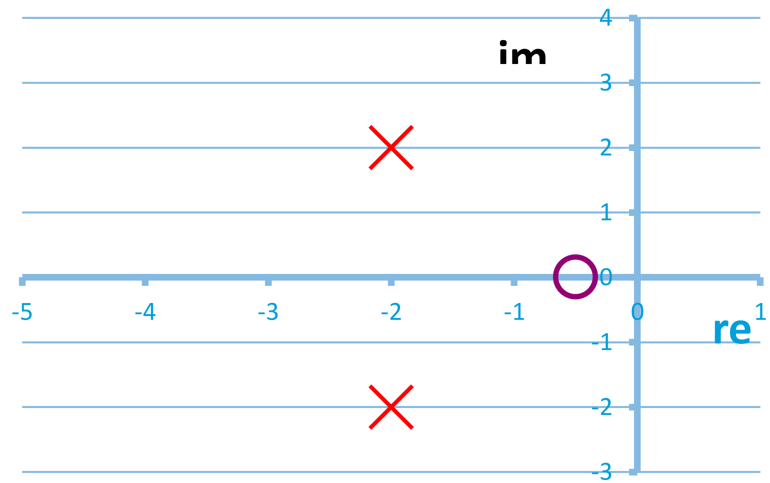
WHERE STUDENTS MATTER

Recap

We have learned to derive the transfer function of a system:

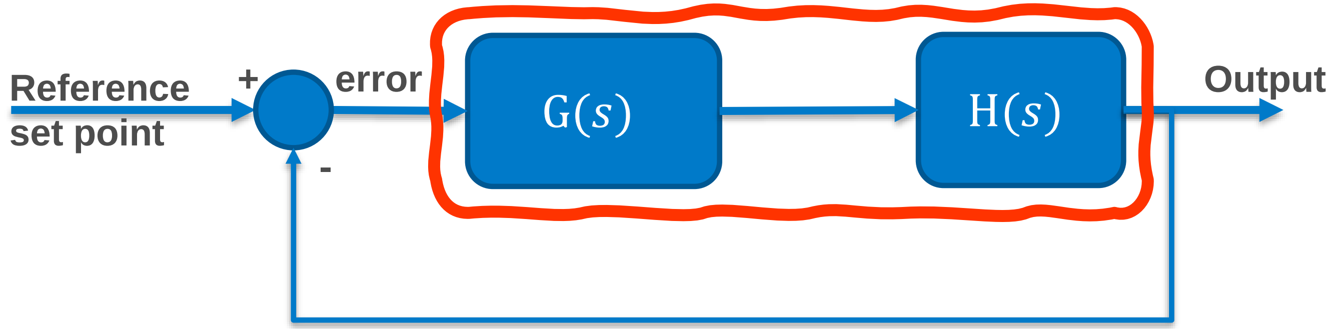
$$H(s) = \frac{2s+1}{s^2+4s+8}$$

We have learned to place the <u>open loop</u> poles and zeros in the s-plane



Next step:
learn about the stability of the system by plotting /drawing the root locus and see the location of the <u>closed loop</u> poles

CLOSED LOOP TF



We were working with open loop transfer function

$$G(s)H(s) = K\frac{N(s)}{D(s)}$$

The closed loop transfer function:

$$\frac{G(s)H(s)}{1 + G(s)H(s)}$$

CLOSED LOOP TF
POLES AND ZEROS

We were working with open loop transfer function

$$OLTF = G(s)H(s) = K\frac{N(s)}{D(s)}$$

The closed loop transfer function:

$$CLTF = \frac{G(s)H(s)}{1 + G(s)H(s)}$$

What happens to poles and zeros of the CLTF?

CLOSED LOOP TF POLES AND ZEROS

We were working with open loop transfer function

$$OLTF = G(s)H(s) = K\frac{N(s)}{D(s)}$$

The closed loop transfer function:

$$CLTF = \frac{G(s)H(s)}{1+G(s)H(s)} = \frac{K\frac{N(s)}{D(s)}}{1+K\frac{N(s)}{D(s)}} = \frac{KN(s)}{D(s)+KN(s)}$$

POLES MOVE!
ZEROS STAYS!

EXAMPLE:
SECOND-ORDER OLTF

$$H(s) = \frac{K}{as^2 + bs + c}$$

$$as^2 + bs + c = 0$$

$$p_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

EXAMPLE: SECOND-ORDER CLTF

$$H(s) = \frac{K}{as^2 + bs + (c + K)}$$

We may have a simple gain controller

$$as^2 + bs + (c + K) = 0$$

$$p_{1,2} = \frac{-b \pm \sqrt{b^2 - 4a(c + K)}}{2a}$$

$$= \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4a(c + K)}}{2a}$$

EXAMPLE: SECOND-ORDER CLTF

$$p_{1,2} = \frac{-b \pm \sqrt{b^2 - 4a(c+K)}}{2a} = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4a(c+K)}}{2a}$$

We start with two real OLTF poles.
We have $b^2 > 4ac$. We increase K.

When K = 0, we obtain the OLTF poles as the CLTF poles.

As K ↑ , the CLTF poles tends to converge to 2 identical poles at $\frac{-b}{2a}$ till $b^2 = 4a(c+K)$.

Further as K ↑, the real part of the poles stays the same at $\frac{-b}{2a}$. The imaginary part of the poles appear and diverges.

STABILITY!

Position of poles determines the stability!

When we have a closed loop system with a controller, the poles of the CLTF moves away from the poles of OLTF!

To determine the stability of the entire control system, we need to know the position of poles of the closed loop system.

BACK TO OUR EXAMPLE: SECOND-ORDER CLTF

If we are given an OLTF, we can draw the poles when the gain controller is 0.

But we can also infer how the CLTF poles changes when we tune the controller gain larger!

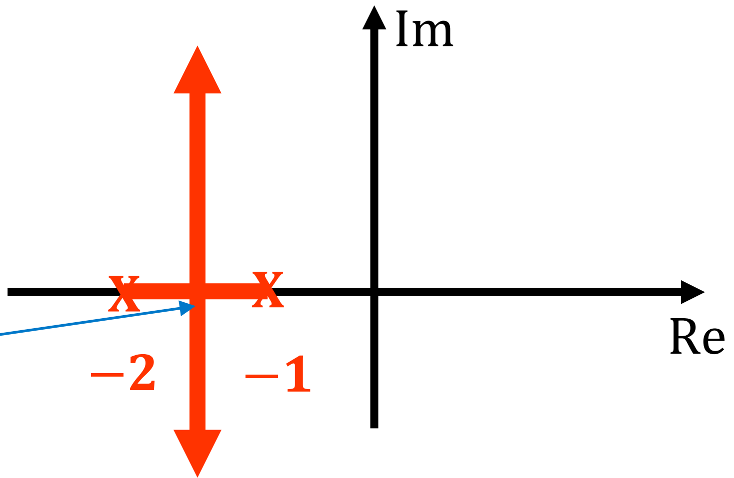
The poles moves in continuous trajectories.
We can easily draw them:

OLTF characteristic eq:

$$(s+1)(s+2)$$
$$= s^2 + 3s + 2$$

CLTF characteristic eq:

$$s^2 + 3s + (2+K)$$
$$\frac{-b}{2a} = \frac{-3}{2}$$



BACK TO OUR EXAMPLE: SECOND-ORDER CLTF

If we are given an OLTF, we can draw the poles when the gain controller is 0.
But we can also infer how the CLTF poles changes when we tune the controller gain larger!
The poles moves in continuous trajectories.
We can easily draw them:

We just draw the "**<u>Root Locus</u>**"

OLTF characteristic eq:

$$(s+1)(s+2)$$
$$= s^2 + 3s + 2$$

CLTF characteristic eq:

$$s^2 + 3s + (2+K)$$
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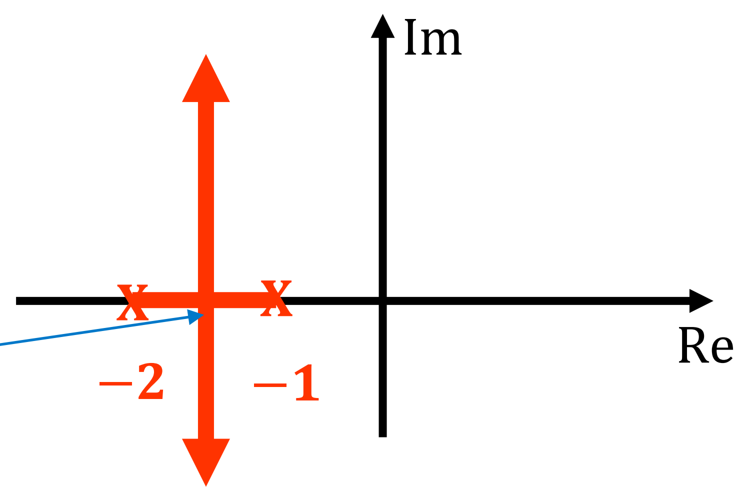
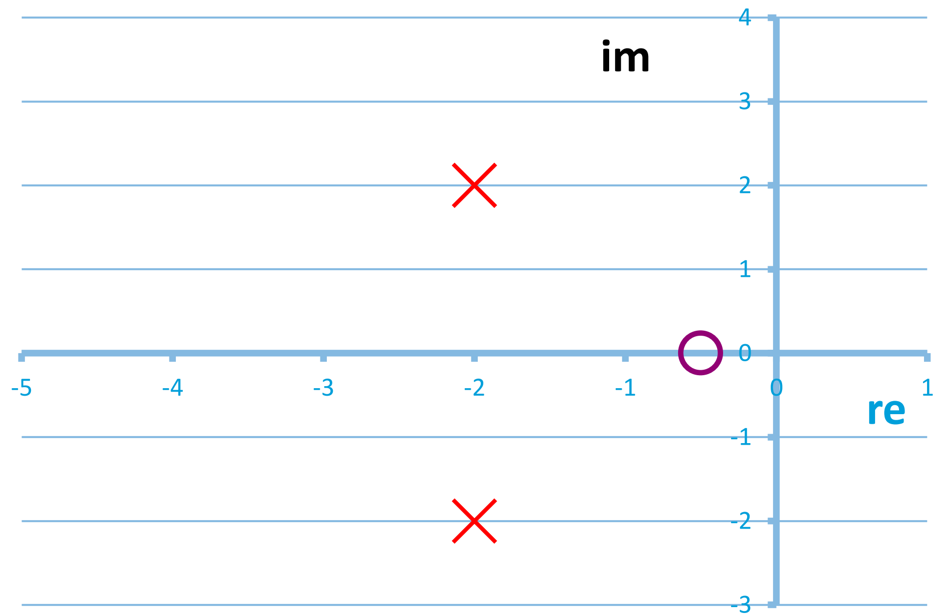


What is the root locus about?

The root locus plots the poles of the closed loop transfer function in the complex *s-plane* as a function of a gain parameter K

Open loop poles:

Up until now we have drawn the poles and zero in the s-plane and reported the gain of the transfer function in the plot.



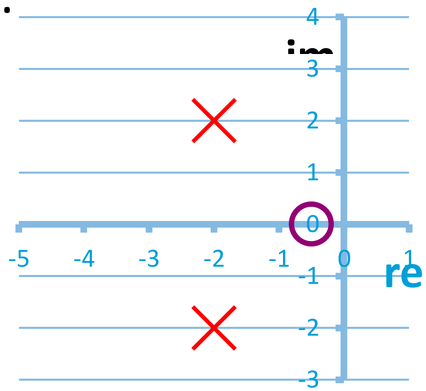
What is the root locus about?

The root locus plots the <u>poles</u> of the <u>closed loop transfer function</u> in the complex <u>*s*-plane</u> as a function of a gain parameter K

<u>Open loop poles:</u>

Up until now we have drawn the poles and zero in the s-plane and reported the gain of the transfer function in the plot.

P = -2 + 2j and -2-2j , Z = -0.5



OLTF:

$$\frac{2K(s+\frac{1}{2})}{(s-2+2j)\,(s-2-2j)}$$

CLTF:

$$\frac{2K(s+\frac{1}{2})}{(s-2+2j)\,(s-2-2j)+2K(s+\frac{1}{2})}$$

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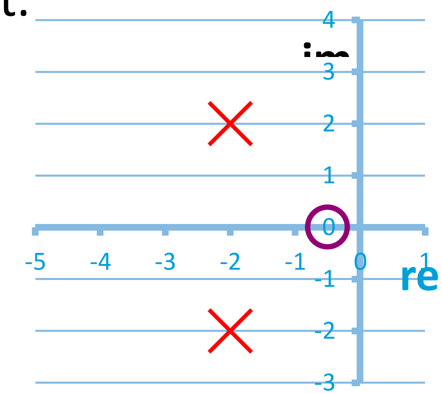


OLTF:

$$\frac{2K(s+\frac{1}{2})}{(s-2+2j)\,(s-2-2j)}$$

CLTF:

$$\frac{2K(s+\frac{1}{2})}{(s-2+2j)\,(s-2-2j)+\boxed{2K(s+\frac{1}{2})}}$$

When K small, OLTF poles dominate the denominator.

When K very large, $\boxed{\text{OLTF zeros}}$ dominate the denominator.

What is the root locus about?

The root locus plots the poles of the closed loop transfer function in the complex s-plane as a function of a gain parameter K

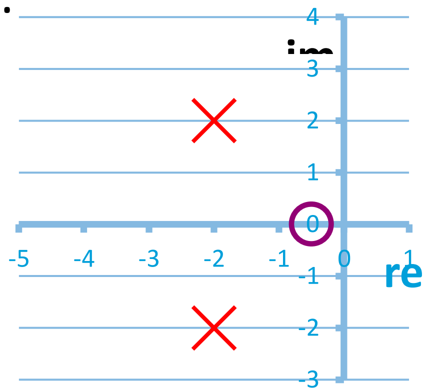
Open loop poles:

Up until now we have drawn the poles and zero in the s-plane and reported the gain of the transfer function in the plot.

P = -2 + 2j and -2-2j , Z = -0.5

OLTF with controller K:

$$\frac{2K(s+\frac{1}{2})}{(s-2+2j)\,(s-2-2j)}$$



CLTF:

$$\frac{2K(s+\frac{1}{2})}{(s-2+2j)\,(s-2-2j)+\boxed{2K(s+\frac{1}{2})}}$$

As K increase from very small to very large, the CLTF poles are moving from OLTF poles to OLTF zeros

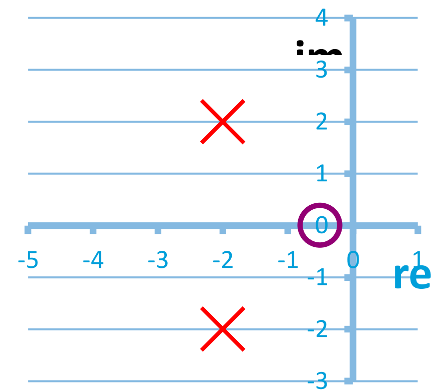
What is the root locus about?

The root locus plots the poles of the closed loop transfer function in the complex s-plane as a function of a gain parameter K

CLTF:

$$\frac{2Ks+K}{s^2+4s+8+2Ks+K}$$

$$\frac{2Ks+K}{s^2+(4+2K)s+(8+K)}$$



Poles: $p = -(2+K) \pm \frac{\sqrt{(4+2K)^2-4(8+K)}}{2} = -(2+K) \pm \sqrt{K^2+3K-4}$

$$K^2+2K-4 = (K+4)(K-1)$$

When $-4 < 0 < K < 1$, we have $K^2+2K-4 < 0$.

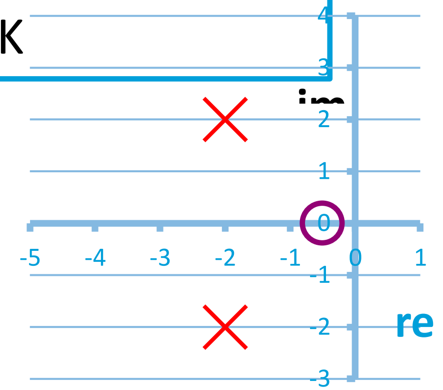
As K = 1, the imaginary part vanishes.

A. $\lim\limits_{K\to+\infty} -(2+K) - \sqrt{K^2+3K-4} = -\infty$ **(Obvious)**

B. $\lim\limits_{K\to+\infty} -(2+K) + \sqrt{K^2+3K-4}$

What is the root locus about?

The root locus plots the poles of the closed loop transfer function in the complex *s*-plane as a function of a gain parameter K



A. $\lim\limits_{K\to+\infty} -(2+K)-\sqrt{K^2+3K-4} = -\infty$

B. $\lim\limits_{K\to+\infty} -(2+K)+\sqrt{K^2+3K-4}$

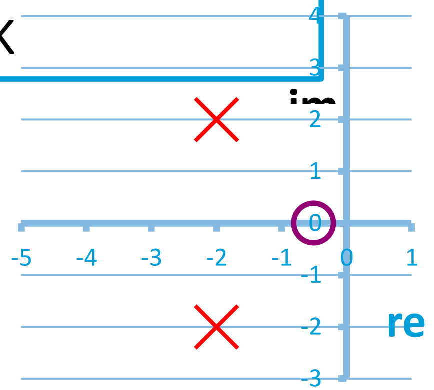
$$= \lim_{K\to+\infty} \frac{\left((2+K)+\sqrt{K^2+3K-4}\right)\left(-(2+K)+\sqrt{K^2+3K-4}\right)}{(2+K)+\sqrt{K^2+3K-4}}$$

$$= \lim_{K\to+\infty} \frac{-K}{(2+K)+\sqrt{K^2+3K-4}}$$

$$= \lim_{K\to+\infty} \frac{-1}{\frac{2}{K}+1+\sqrt{1+\frac{3}{K}-\frac{4}{K^2}}}$$

What is the root locus about?

The root locus plots the poles of the closed loop transfer function in the complex s-plane as a function of a gain parameter K



A. $\lim_{K\to+\infty} -(2+K) - \sqrt{K^2+3K-4} = -\infty$

B. $\lim_{K\to+\infty} -(2+K) + \sqrt{K^2+3K-4}$

$$= \lim_{K\to+\infty} \frac{\left((2+K)+\sqrt{K^2+3K-4}\right)\left(-(2+K)+\sqrt{K^2+3K-4}\right)}{(2+K)+\sqrt{K^2+3K-4}}$$

$$= \lim_{K\to+\infty} \frac{-K}{(2+K)+\sqrt{K^2+3K-4}}$$

$$= \lim_{K\to+\infty} \frac{-1}{\cancel{\frac{2}{K}}+1+\sqrt{1+\cancel{\frac{3}{K}}-\cancel{\frac{4}{K^2}}}}$$

$$= -\frac{1}{2}$$

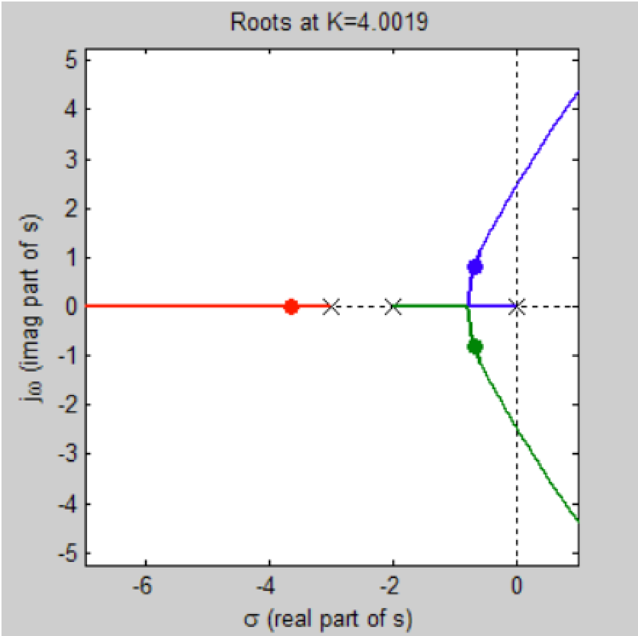
This is our zero!

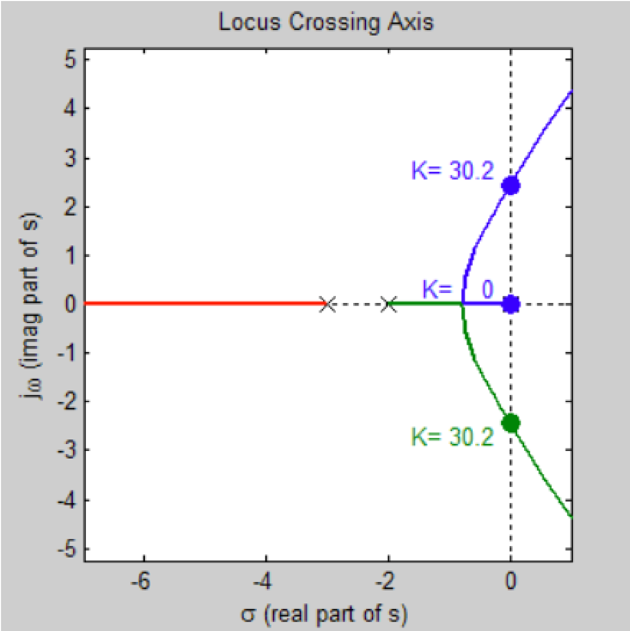
What is the root locus about?

The root locus plots the poles of the closed loop transfer function in the complex *s*-plane as a function of a gain parameter

Closed loop poles:

The gain in the transfer function influences the position of the poles. The way the poles move as the gain increases form 0 to actual gain in the system is represented by the root locus.

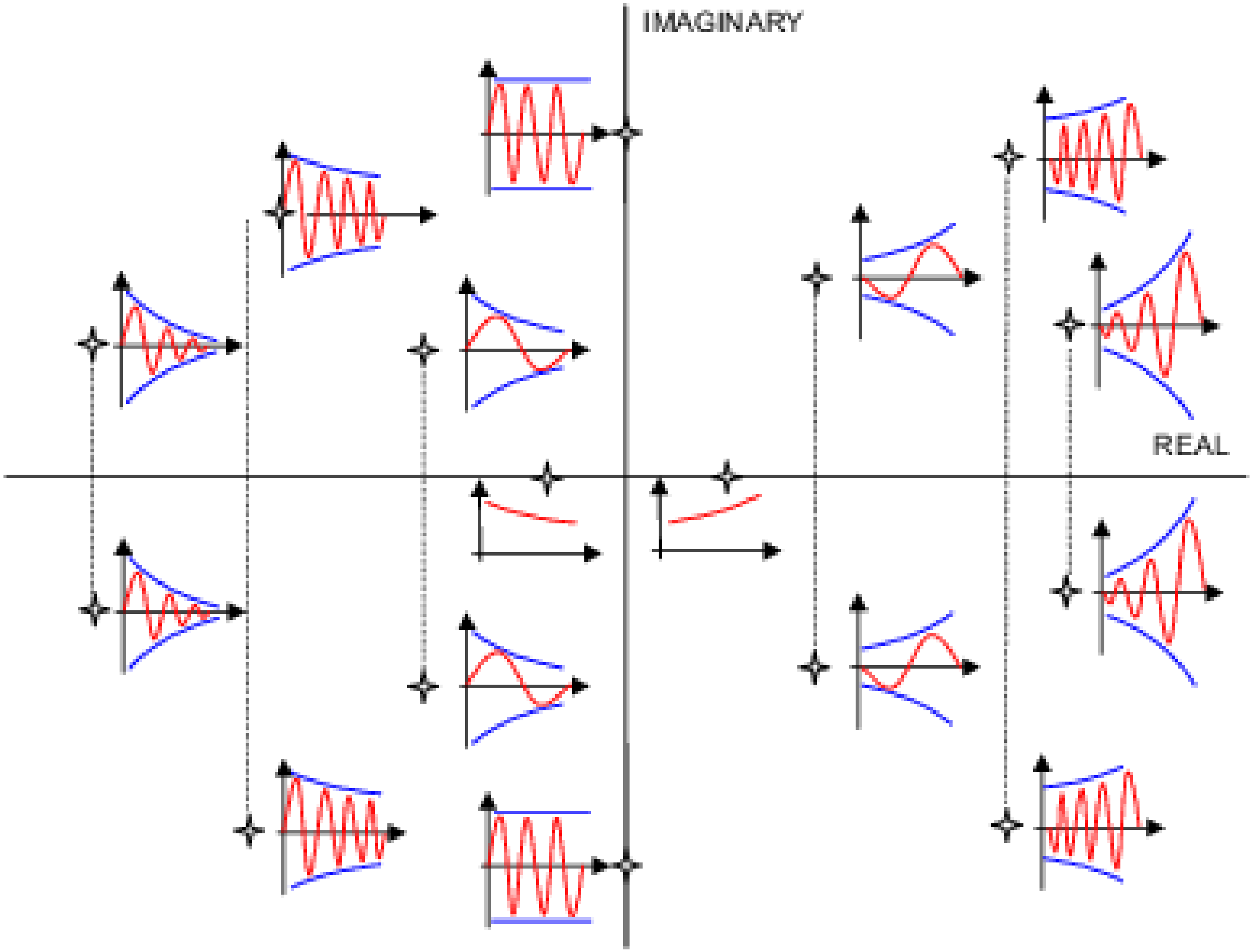




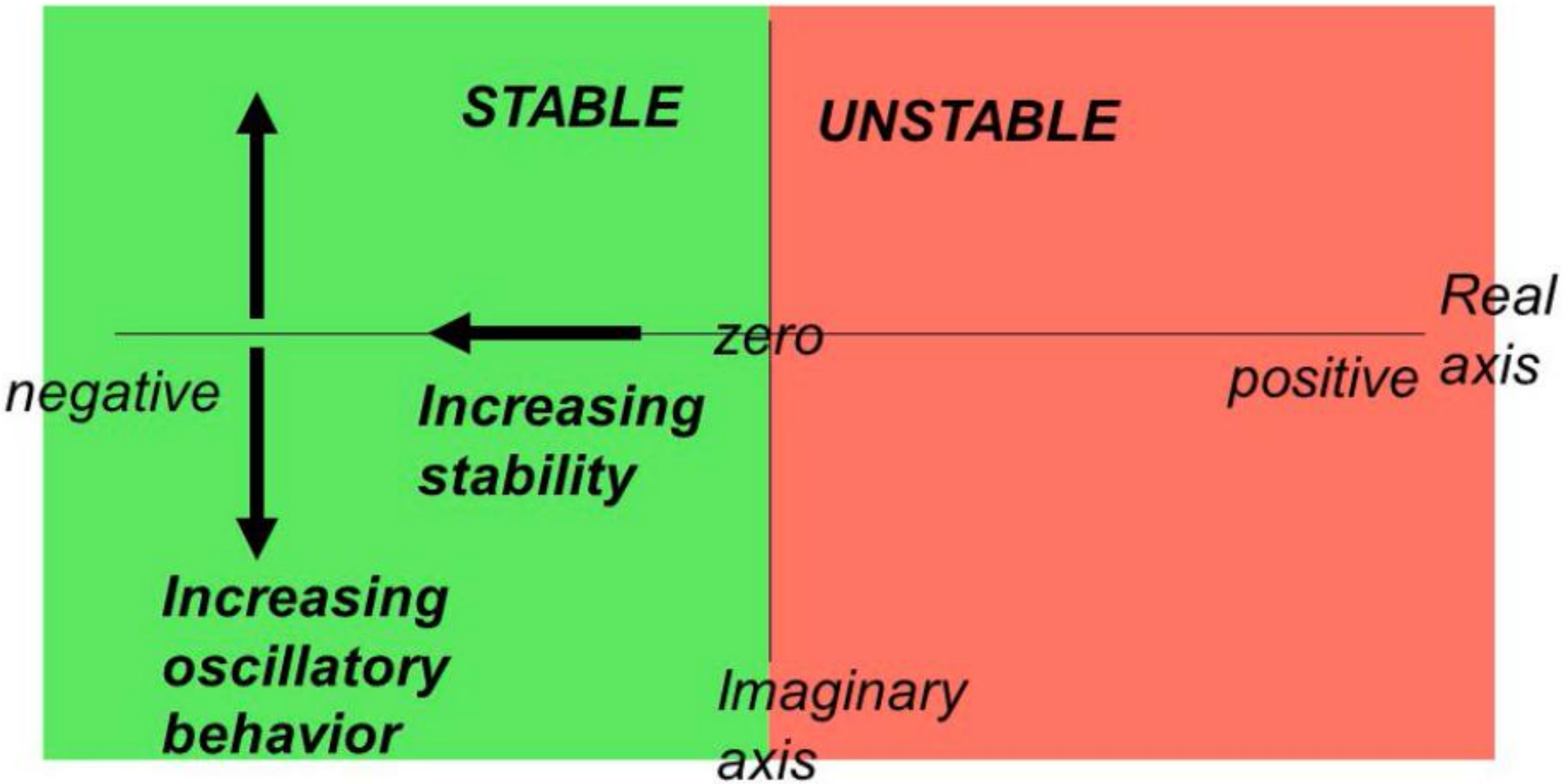
The system behaviour is determined by the position of the closed loop poles.

Pole location information (graphical)

The impulse response for poles at different location in the s-plane

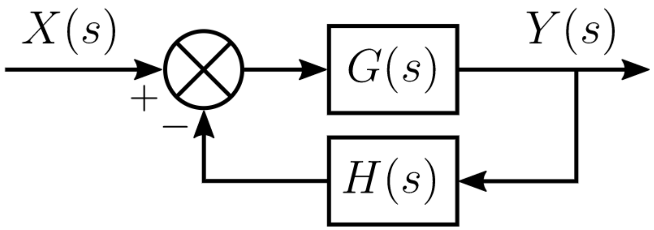


Root locus and stability



Characteristic equation

System



Closed loop transfer function

$$T(s) = \frac{Y(s)}{X(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

The characteristic equation is defined by setting the denominator of the closed loop transfer function to zero.

The characteristic equation: $1 + G(s)H(s) = 0$

The product *G(s)H(s)* can often be expressed as a rational polynomial function:

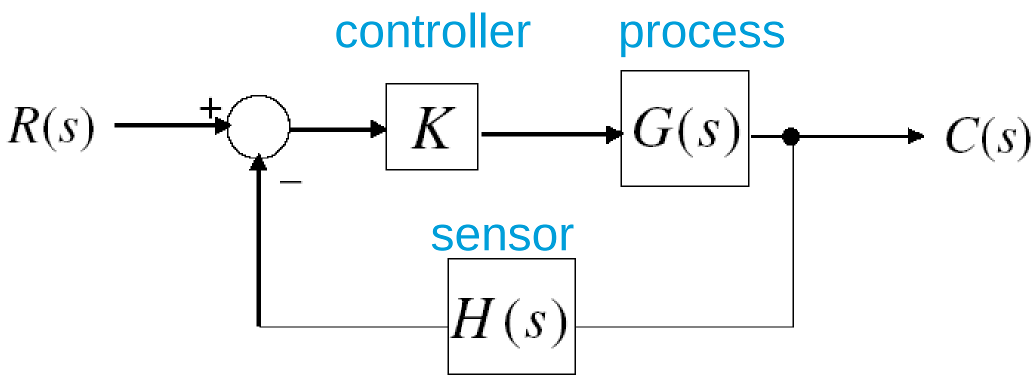
$$G(s)H(s) = K\frac{(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

Where *K* is gain, $-z_n$ are zeros and $-p_n$ are poles.

The value of *K* does not affect the location of the zeros.
The open-loop zeros are the same as the closed-loop zeros

Root locus example

- General system for root locus



  - For example consider process $G(s) = \dfrac{5}{(s+1)}$
  - with a sensor $H(s) = 1$
  - Furthermore we consider only nonnegative *K*, so $K \geq 0$

Root locus example

The ‘pzmap’ of the process shows the location of its zeros and poles

The process: $$G(s) = \frac{5}{(s+1)}$$

Zeros:

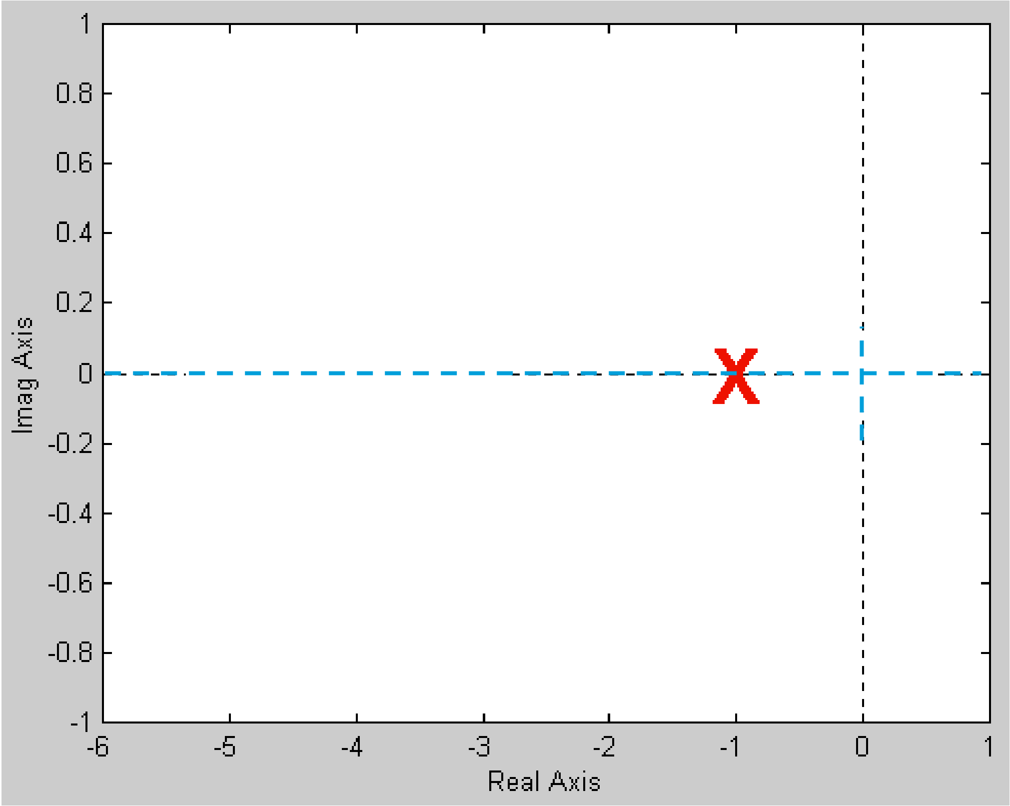
numerator = 0

no zero

Poles:

denominator = 0

s = -1

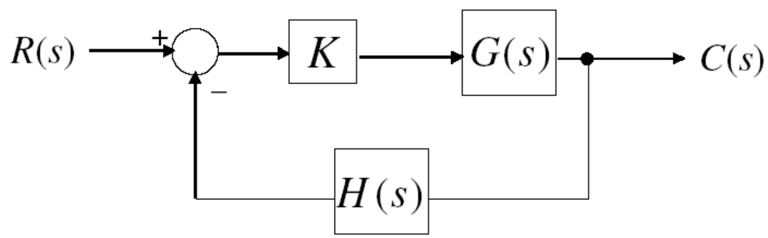


In Matlab

```
>> sys=tf(5,[1 1])
>> pzmap(sys)
```

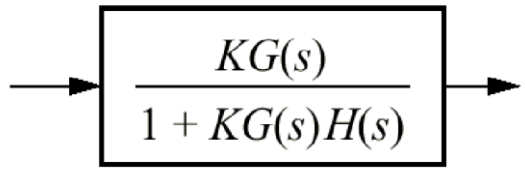
Root locus example

What are the values of the zeros and poles of the closed loop?



$$G(s) = \frac{5}{(s+1)}$$

$$H(s) = 1$$



Closed loop TF:

$$H_{cl}(s) = \frac{5K}{(s+1)+5K}$$

Zeros: set numerator = 0

here no zeros, but else independent on value of K

Poles: set denominator = 0

$$s + (5K+1) = 0$$

the poles are dependent on value of K

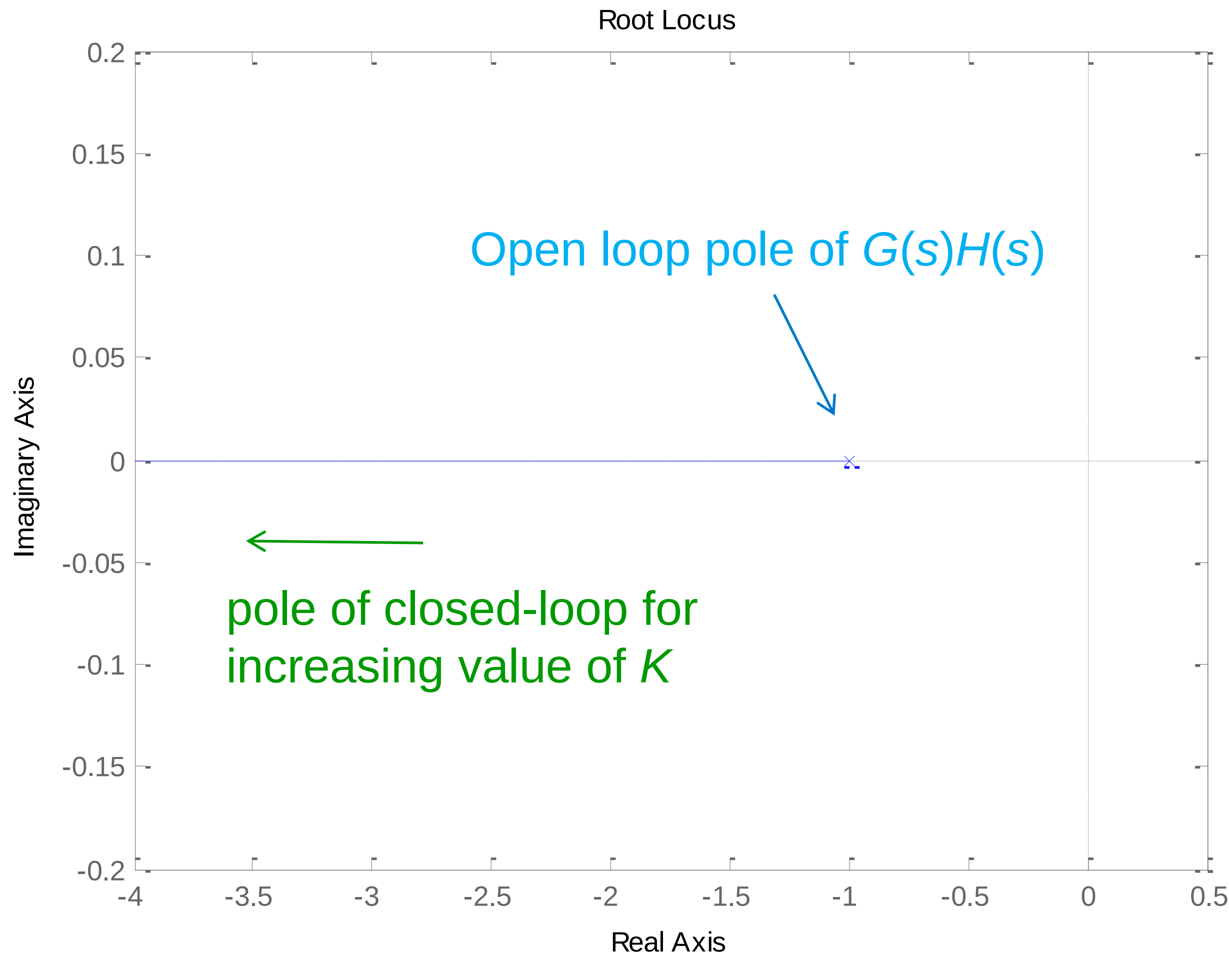
$$\boxed{s = -5K - 1}$$

Root locus example

$$G(s)H(s)=\frac{5}{(s+1)}$$

In Matlab

```
>> sys=tf(5,[1 1])
>> rlocus(sys)
or
> > rltool(sys)
```



For which value of K is the closed-loop stable?

Starts at K=0 → s=-1, with increasing K → s remains negative, so always stable

Root locus example of using Matlab

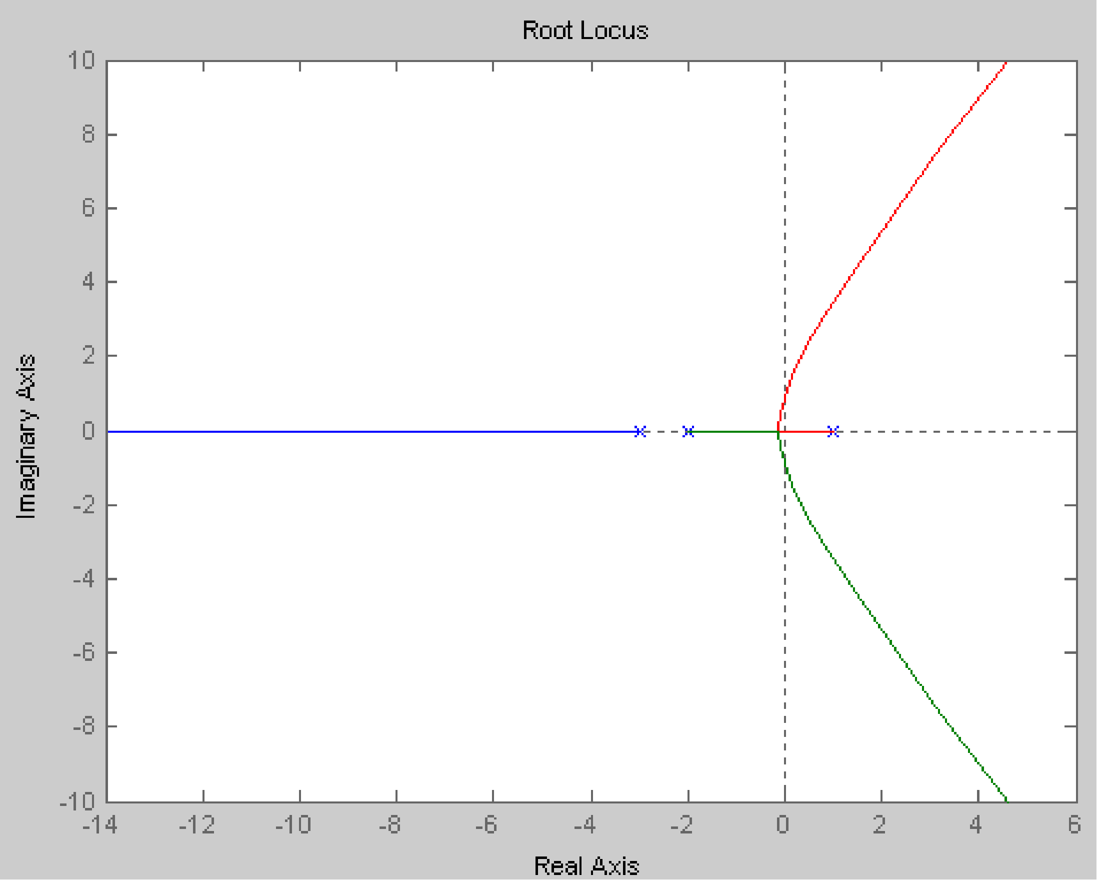
$$G(s)H(s) = \frac{35}{s^3 + 4s^2 + s - 6}$$

Matlab:

```
>>num=[35];
>>den=[1 4 1 -6];
>>sys=tf(num,den);
>>rlocus(sys)
```

or use

```
>>rltool(sys)
```



ROOT-LOCUS ASYMPTOTES

Intuition:

$$CLTF = \frac{G(s)H(s)}{1 + G(s)H(s)} = \frac{K\dfrac{N(s)}{D(s)}}{1 + K\dfrac{N(s)}{D(s)}} = \frac{KN(s)}{D(s) + KN(s)}$$

CLTF characteristic equation:

$$D(s) + KN(s) = 0$$

$$\frac{D(s)}{N(s)} = \frac{(s - p_1)(s - p_2)(s - p_3)\,\dots}{(s - z_1)(s - z_2)(s - z_3)\,\dots} = -K$$

As $K \rightarrow \infty$, we can approximately reduce the above equation to:

$$s^n = -\infty, \qquad \text{with } n = \#P - \#Z$$

ROOT-LOCUS ASYMPTOTES

$$s^n = -\infty, \qquad \text{with } n = \#P - \#Z$$

We would like to solve for s, thus we have:

$$s = (-\infty)^{\frac{1}{n}}$$

If we write both sides of the equation in polar form:

$$|s|e^{i\theta} = |\infty|^{\frac{1}{n}} e^{i\frac{(\pi + 2k\pi)}{n}}, \qquad k \in \mathbb{Z}$$

We can see that

$$\theta = \frac{(\pi + 2k\pi)}{n}, \qquad k \in \mathbb{Z}$$

ROOT-LOCUS ASYMPTOTES

Looking back:

CLTF characteristic equation:

$$D(s) + KN(s) = 0$$

$$\frac{D(s)}{N(s)} = \frac{(s-p_1)(s-p_2)(s-p_3)\ldots}{(s-z_1)(s-z_2)(s-z_3)\ldots} = -K$$

Let $\alpha = \#P$ and $\beta = \#Z$,approximating polynomials using the two highest order terms:

$$\frac{(s-p_1)(s-p_2)(s-p_3)\ldots}{(s-z_1)(s-z_2)(s-z_3)\ldots} \sim \frac{s^{\alpha} + (\sum p)\, s^{\alpha-1}}{s^{\beta} + (\sum z)\, s^{\beta-1}}$$

ROOT-LOCUS ASYMPTOTES

$$\frac{(s-p_1)(s-p_2)(s-p_3)\ldots}{(s-z_1)(s-z_2)(s-z_3)\ldots} \sim \frac{s^\alpha + (\sum p)\, s^{\alpha-1}}{s^\beta + (\sum z)\, s^{\beta-1}}$$

We can assume pole-zero cancellation. We can most safely do that when we assume $k \to \infty$ and that is what we were calculating. Apply polynomial long division leads us to:

$$\frac{s^\alpha + (\sum p)\, s^{\alpha-1}}{s^\beta + (\sum z)\, s^{\beta-1}} \sim \frac{1}{s^{\beta-\alpha} + (\sum p - \sum z)\, s^{\beta-\alpha-1}}$$

$$s^{\beta-\alpha} + (\sum p - \sum z) s^{\beta-\alpha-1} \sim \left(s - \frac{\sum p - \sum z}{\beta - \alpha}\right)^{\beta-\alpha}$$

We have found the centroid of our asymptotes!

$$\frac{\sum p - \sum z}{\#P - \#Z}$$

ROOT-LOCUS ASYMPTOTES

The root-locus asymptotes need to symmetrically scan through the entire unit circle.

Number of asymptotes: $\#P - \#Z$

Angles separation between asymptotes next to each other:

$$\frac{360^{o}}{\#P \ - \#Z}$$

For k-th asymptote:

$$\frac{360^{o}k + 180^{o}}{\#P \ - \#Z}$$

CLTF REPEATED ROOTS WHY $-\frac{b}{2a}$?

Departure point (when this polynomial contains repeated roots).

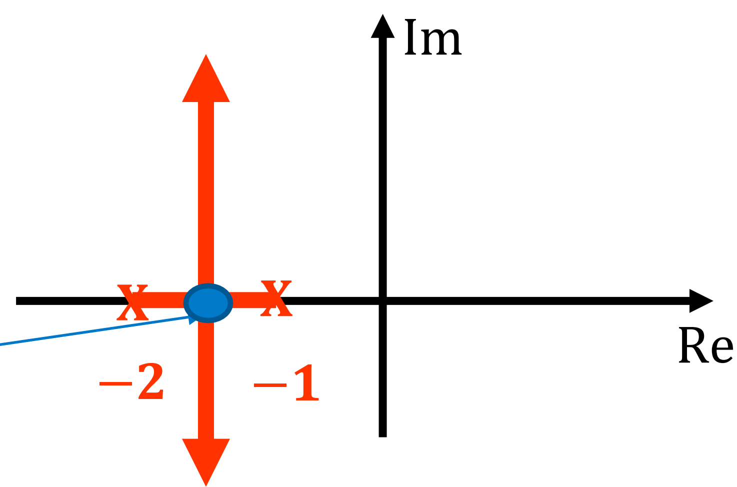
Theorem:
A polynomial $p(x)$ with real coefficients has a root r repeated n times, then the first n derivatives of $p(x)$ at that root r will be zero.

OLTF characteristic eq:

$$(s+1)(s+2)$$
$$= s^2 + 3s + 2$$

CLTF characteristic eq:

$$s^2 + 3s + (2+K)$$
$$\frac{-b}{2a} = \frac{-3}{2}$$



CLTF REPEATED ROOTS
WHY $-\frac{b}{2a}$?

Theorem:
A polynomial $p(x)$ with real coefficients has a root r with multiplicity n, then the first $n-1$ derivatives of $p(x)$ at that root r will be zero.

Back to our 2nd order characteristic equation:

$$\rho(s) = as^2 + bs + c = 0$$

If we know $s = r$ is a double root (multiplicity 2), then:

$$\rho(r) = 0$$
$$\rho'(r) = 0$$

Thus we can derive:

$$ar^2 + br + c = 0$$
$$2ar + b = 0$$

$$r = -\frac{b}{2a}$$

ROOT-LOCUS CONVENIENT RULES

1. Where does the root locus start and end?

   Start at poles of OLTF, ends at finite zeros or infinity of OLTF. Number of branches = number of poles
2. Where is the locus on the real axis?

   To the left of an odd number of real axis poles & zeros
3. What is the shape of root locus?

   $$\beta = \frac{\sum p - \sum z}{\#p - \#n} \text{ (Centroid of asymptotes)}$$

   $$\phi_l = \frac{(2l+1)180^o}{\#p - \#n}, \mathrm{l} = 0,1, \ldots (\#p - \#n - 1)$$

   (Angles of asymptotes)

   The root locus is symmetric about the real axis!
4. Where does the root locus break in or out?

   Solve: $\frac{\mathrm{d}\,(1+KL(s))}{\mathrm{d}\,s} = 0$

Root locus

how to determine manually stability/instability?

1. Establish characteristic equation

2. Replace s by $j\omega$ *(The boundary condition for instability is at the imaginary axis ≡ real (σ) is zero)*

3. Sum of real part is 0

4. Sum of imaginary part is 0

5. The maximum allowable K can now be calculated

Root locus

how to determine manually stability/instability?

$$G(s)H(s)=\frac{10s^2-4s+1}{s^2+2s}$$

1. Establish root locus

$$s^2+2s+K(10s^2-4s+1)=0$$
$$(10K+1)s^2+(2-4K)s+K=0$$

Root locus

how to determine manually stability/instability?

$$G(s)H(s) = \frac{10s^2 - 4s + 1}{s^2 + 2s}$$

1. Establish root locus

$$s^2 + 2s + K(10s^2 - 4s + 1) = 0$$

$$(10K + 1)s^2 + (2 - 4K)s + K = 0$$

2. Replace s by jω

$$(10K + 1)(j\omega)^2 + (2 - 4K)j\omega + K = 0$$

$$(2 - 4K)j\omega + K - (10K + 1)\omega^2 = 0$$

Root locus

how to determine manually stability/instability?

$$G(s)H(s)=\frac{10s^2-4s+1}{s^2+2s}$$

1. Establish root locus
$$s^2+2s+K(10s^2-4s+1)=0$$
$$(10K+1)s^2+(2-4K)s+K=0$$

2. Replace s by jω
$$(10K+1)(j\omega)^2+(2-4K)j\omega+K=0$$
$$(2-4K)j\omega+K-(10K+1)\omega^2=0$$

3. Sum of real part is 0

Root locus

how to determine manually stability/instability?

$$G(s)H(s) = \frac{10s^2 - 4s + 1}{s^2 + 2s}$$

1. Establish root locus

$$s^2 + 2s + K(10s^2 - 4s + 1) = 0$$

$$(10K + 1)s^2 + (2 - 4K)s + K = 0$$

2. Replace s by jω

$$(10K + 1)(j\omega)^2 + (2 - 4K)j\omega + K = 0$$

$$(2 - 4K)j\omega + K - (10K + 1)\omega^2 = 0$$

3. Sum of real part is 0

$$K - (10K + 1)\omega^2 = 0$$

$$\Rightarrow \omega^2 = \frac{K}{(10K + 1)}$$

Root locus

how to determine manually stability/instability?

$$G(s)H(s) = \frac{10s^2 - 4s + 1}{s^2 + 2s}$$

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4. Sum of the imaginary part is 0

Root locus

how to determine manually stability/instability?

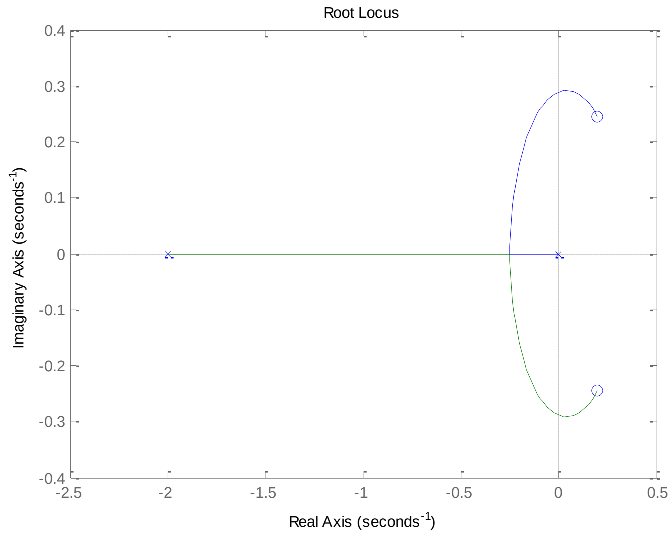
$$G(s)H(s)=\frac{10s^2-4s+1}{s^2+2s}$$

4. Sum of imaginary part is 0

$$(2-4K)\omega=0$$

$$\Rightarrow K=\frac{2}{4}=0.5$$

5. The maximum allowable *K* can now be calculated



HOMEWORK

Stage TWO Exercises:

- Problem 2
- Problem 5
- Problem 6
- Problem 8
- Problem 9
- Problem 10