PART 1 | FUNDAMENTAL CONCEPTS: CONTROL SYSTEMS

Control systems are an essential part of modern engineering. They help us automate and regulate machines, processes, and devices in our daily lives and industrial environments. Without control systems, many technologies we rely on would not function as smoothly or efficiently.

In this section, we shall introduce the most fundamental concepts that act as stepping stones to build-up our knowledge in the field of classical control systems theory.

1.1 Classical Control Systems

We have a 3-word title: "Classical Control Systems". Easily, following the most primitive intuition, we can break this title down into a couple of questions for clearification:

1. What is control?
2. What is a system?
3. What is a control system?
4. What are *classical* control systems about?

1.1 .1 What is control?

Intuitively, we refer to **control** as the act of achieving what we desire. What we desire can be a numerical value of a physical quantity or a quantifiable state, such as obtaining 220V from an electrical socket, maintaining the room temperature around 23 degrees Celsius, or driving a car in a straight line.

Figure 1.1: Jack is driving.

In Fig. 1.1, Jack, our imaginary character, is driving his car. Jack wants to go straight and maintain his direction. Therefore, he is steadily holding the steering wheel. Occasionally, he needs to steer slightly to maintain the direction due to a bump in the road or misaligned wheel configuration. With his steering wheel control, Jack's car is still going straight as he desires. Jack is in control of his car's direction.

1.1 .2 What is a system?

Almost everything can be a system!

In the context of control systems engineering, what can be regarded as an interesting system depends on the intended control objective.

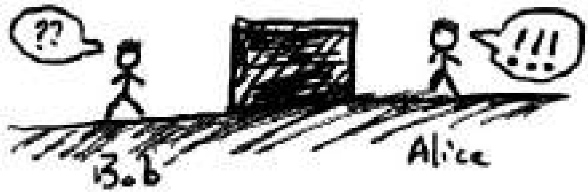


Figure 1.2: The presence of a mysterious metal block.

In Fig. 1.2, two imaginary characters, Alice and Bob, are walking down the road. They are blocked by a mysterious metal block on the ground. Bob is thinking about his vacation in the summer; obviously, this large piece of metal is not particularly interesting to him, and he has no intention of interacting with it. Hence, this mysterious piece of metal is definitely not an interesting system for Bob.

Alice, however, is thinking about building a heat exchanger for an engineering project in her factory. This mysterious piece of metal might just be good material for the heat exchanger! It could be an interesting system for Alice to study, particularly in terms of its specific heat capacity. She might even test it on the spot and gather some data. This mysterious piece of metal can be a very interesting system for Alice.

To summarize, whether something can be regarded as an interesting system depends on the control scenario and what the control systems engineer desires from the system. If something aligns with your interest, it can be regarded as a system to be studied!

1.1 .3 What is a control system?

Combining what we have discussed in the previous sections, a control system is a set of engineered devices or systems that manage, command, direct, or regulate the behavior of other devices or systems. It aims to achieve a desired output by manipulating the flow of signals.

Unless particularly specified, the **input** of a control system should be the desired output state or **the reference point**. **The output** is **the actual performance of the system**. Ideally speaking, if the system is functional as we intended, the output should match the desired state, which is our input. This control system description can fit from a collective large scale system that are composed of multiple sub-systems to a small sub-system that can be regarded as a stand-alone control system.

For example, consider a simple home heating system. You set a desired temperature, and the system turns the heater on or off to maintain that temperature. This is a basic control system.

Illustrated in our example in Fig. 1.1, the desired output for Jack is that the car he is driving continues to move in a straight line. He is manipulating the steering wheel to achieve this desired result. The steering angles can be regarded as a flow of signals. If we look closer, we might observe how the initial *'signal'* traverses through the internal mechanisms of the car and ultimately affects the mechanical angle of the wheels, thereby maintaining the direction of the vehicle. The output of this control system is eventually how the car behaves.

Control systems engineering primarily treats system dynamic behavior, thus we study the *changes* in the system output related to the input. Naturally, when talking about changes of a system over a physical quantity (mostly this is time in control systems), we can obtain a mathematically models that describe such dynamics, especially utilizing differential equations.

1.1 .4 *What are* "classical" *control systems about?*

Classical control is called *classical* because it represents the earliest systematic approach to control system design, developed before the rise of modern control techniques like state-space methods, model predictive control(MPC), fuzzy logic control, and control algorithms using mechine learning techniques, etc. Despite its age, classical control remains widely used due to its simplicity, robustness, intuitive graphical tools, and effectiveness in many practical applications. In addition, classical control systems often requires a precise system model, which often require specific and accurate description of physical processes. This makes classical control systems theory a solid fundation of the more advanced techniques.

Classical control systems deals with:

causal ***s****ingle-****i****nput-****s****ingle-****o****utput(SISO)* ***l****inear* ***t****ime-****i****nvariant*(LTI)

systems.

We shall look at these properties one by one.

1.1 .5 *Causality: cause before effect*

A causal system does not anticipate the future input. Its reaction is only based on the history and/or the present.

Definition 1 (Causal System). A system $\mathcal{S}$ described by an input-output relationship $x(t) \xrightarrow{\mathcal{S}} y(t)$ is *causal* if, for any time t_0, the output $y(t)$ at time t_0 depends only on the input $x(t)$ for $t \leqslant t_0$.

$$y(t_0) = \mathcal{S}(x(t)) \quad \forall t \leqslant t_0$$

If the output $y(t)$ at time $t = t_0$ depends on inputs $x(t)$ that span among the past, and/or the present, and the future: $t \in \mathbb{R}$, then the system is non-causal. If the output $y(t)$ at time $t = t_0$ only depends on future inputs $x(t)$ where $t > t_0$, the system is anti-causal.

Consider a home heating control system, the differences between causal, non-causal, and anti-causal systems are as follows:

- **Causal system:** A home heating system adjusts the temperature based on the current and past readings. It does *not* know the future temperature; it reacts only when the room gets too cold or too hot.
- **Non-causal system:** A system that turns on the heater today because it knows it will be very cold tomorrow comparing to the temperature today. This kind of control requires knowledge of the future, which is impossible in real-time systems.
- **Anti-causal system:** A system that turns on the heater today because it knows it will be very cold tomorrow irrespective of the current temperature. This kind of control *only* requires knowledge of the future, *irrespective* of the present and the past. Anti-causal systems are impossible and not practical for real-time systems.

Non-causal systems are often used in offline simulations and processing like offline signal processing, image processing, distortion recovery, or predictive models. While anti-causal systems are hypothetical, they recently found their spot in the theoretical research in the field of quantum mechanics.

We shall foucs on causal systems in the following content becuase methods developed for classical control systems are often designed for causal systems.

Moreover, do keep in mind that:

Real-time control systems acting on real physical systems in the real world must always be causal.

1.1.6 Single input single output(SISO) systems

Classical control systems deals with SISO systems that one input signal affects the system and one desired output signal is treated as the system's response. A common representation of the SISO system is:

$$\text{Input}\, x(t) \xrightarrow{\text{System } \mathcal{S}:\, h(t)} \text{Output}\, y(t) \tag{1.1}$$

In time domain description, the output function $y(t)$ is obtained by time domain **convolution** between the input function $x(t)$ and the system's $\mathcal{S}$ impule response function $h(t)$.

$$\begin{aligned} y(t) &= [x * h](t) \\ &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau \end{aligned}$$

If the system $\mathcal{S}$ is causal, the comvolution can be over the non-negative time axis till the present time:

$$y(t) = \int_0^t x(\tau)h(t-\tau)\,d\tau \tag{1.2}$$

In practice, many existing multi-input multi-output(MIMO) systems are hard to analyze. Changes in one sub-system can result in un-desireable change in other interacting sub-systems.

Decoupling is a widely deployed practical approach that can decouple certain MIMO systems into seperated SISO sub-systems to mitigate the mutual-influence problem. There are several existing mathematical tools for decoupling MIMO systems, including singular value decomposition(SVD), relative gain array(RGA), Niederlinski index, decentralized stability condition, etc. If a MIMO system can not be decoupled, then data-driven methods like MPC or neural networks are often deployed.

1.1 .7 Linearity

Definition 2 (Linear System). A system $\mathcal{S} : x(t) \xrightarrow{\mathcal{S}} y(t)$ is a *linear* system if and only if $\mathcal{S}$ satisfies the superposition principle that includes the homogeneity property and additivity property described as follows:

Homogeneity:

$$x(t) \xrightarrow{\mathcal{S}} y(t) \quad \Rightarrow \quad \alpha x(t) \xrightarrow{\mathcal{S}} \alpha y(t), \quad \forall \alpha \in \mathbb{R}\,.$$

Additivity:

$$x_1(t) \xrightarrow{\mathcal{S}} y_1(t)\,, x_2(t) \xrightarrow{\mathcal{S}} y_2(t) \quad \Rightarrow \quad x_1(t) + x_2(t) \xrightarrow{\mathcal{S}} y_1(t) + y_2(t)\,.$$

In total we shall obtain:

$$x_1(t) \xrightarrow{\mathcal{S}} y_1(t)\,, x_2(t) \xrightarrow{\mathcal{S}} y_2(t)$$

$$\Rightarrow$$

$$\alpha x_1(t) + \beta x_2(t) \xrightarrow{\mathcal{S}} \alpha y_1(t) + \beta y_2(t) \quad \forall \alpha, \beta \in \mathbb{R}\,.$$

Linear systems play a crucial role in classical control systems theory. Many techniques we will cover in later sections are primarily designed for linear systems. Even though many real-world systems are non-linear, they can often be linearized using linear approximation for its operating range around an equilibrium point using Taylor series expansion.

1.1.8 Time-invariance

A system is time-invariant means that the response of the system does not depend on the absolute time given the same initial state.

Definition 3 (Time-Invariant System). A system $\mathcal{S}$ described by an input-output relationship $x(t) \xrightarrow{\mathcal{S}} y(t)$ is *time-invariant* if and only if:

$$x(t) \xrightarrow{\mathcal{S}} y(t) \quad \Rightarrow \quad y(t-t_0) \xrightarrow{\mathcal{S}} y(t-t_0), \quad \forall t_0$$

We shall provide an intuitive explanation of time-invariance. In Fig. 1.3, Jack is going down the stairs. Given the same state of Jack and the stairs, the process of Jack going down the stairs is the same. It does not matter if Jack is going down the stairs today, or tomorrow, or at any other time. The process of *Jack going down the stairs* remainds the same, thus the process is time-invariant.

Figure 1.3: Jack on the stairs.

Another practical example would be the heat dissipated through the same resistor that follows the equation $Q(t) = \int_{t_{start}}^{t_{end}} I^2(t) R \, dt$. It does not matter at which time we supply electricity to the resistor, either in the morning or in the afternoon or evening or midnight. As long as it is the same resistor, the same amount of current, and the same duration, the total amount of dissipated heat are the same. The heat dissipation process of a resistor is time-invariant.

1.2 Control loops

Jack wants to practice his balancing skills by standing on one foot. He discovered one phenomenon demonstrated in Fig. 1.4. When Jack closes his eyes, he can only keep his balance for 10 seconds at best. As Jack opens his eyes, he can keep his balance without a problem for a least 2 minutes!

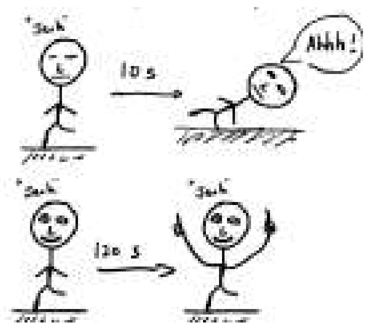


Figure 1.4: Jack standing on one foot.

If we visualize the process when Jack is keeping balance with his eyes closed, we can obtain such a diagram:

$\boxed{\text{input: "keep balance"}} \rightarrow \boxed{\text{Jack}} \rightarrow \boxed{\text{output}}$

Because Jack had his eyes closed, Jack can only rely on his sense of position to balance himself. But Jack has a bad daily posture and reads too much from his phone while keeping his head down, this creates mucle and spine problems that compile errors in Jack's nerve system that distorts his sense of balance. While Jack has his eyes closed, he can not utilize visual reference to calibrate his sense of balance. The errors compile easily and rapidly in his brain, thus Jack loses his balance quickly with his eyes closed.

When Jack opens his eyes, his visual senses provide feedback to his brain about his body position, helping him recalibrate his sense of balance. With the visual *feedback* provided by his eyes, Jack can correct the errors and keep his balance for a prolonged period of time. This process is visualized in Fig. 1.5.

Figure 1.5: Jack keeping balance with visual feedback.

In classical control systems, we typically work with two types of systems:

- systems without feedback, we refer them as: **open-loop systems**,
- systems with feedback, we refer them as: **closed-loop systems**.

Open-loop systems are sometimes called feed-forward systems, there is only one direction for signals to flow. Closed-loop systems not only contain signals flowing from input to output, but there are signals with reverse direction, typically used for comparison and computation with the foward signals for error correction purposes.

Since we are sending signal backwards for comparison using a feedback loop, the sign of the comparison between the forward signal and the feedback signal differentiates the type of feedback loops:

1.2.1 Positive feedback

A positive feedback loop adds up the forward signal and the feedback signal. The feedback signal reinforce the input signal to build a momentum. There are electronic guitars that utilize the positive feedback with amplifiers to increase the amount of distortion and gain to achieve the overdrive sounds. Sometimes positive feedbacks are used in circuit design to create oscillators for specific needs. You may also find positive feedback loops in biological processes like blood clotting.

1.2.2 Negative feedback

A negative feedback loop finds the difference between the forward signal and the feedback signal. If the 2 signals to be compared are the input signal X and the output signal Y, naturally we may obtain the error $E = X - Y$. The error $E = X - Y$ can be interpreted as the difference between the true(Y) and desired(X) output. Based on the error, the feedback loop naturally self-regulate the error with an carefully engineered controller.

A classic example of negative feedback control is the human body's temperature regulation system. The hypothalamus acts as a biological thermostat, continuously monitoring body temperature and triggering corrective actions to maintain homeostasis. If the body temperature rises above normal, mechanisms such as sweating and increased blood flow to the skin activate to dissipate heat. Conversely, if the temperature drops, shivering and vasoconstriction help generate and conserve heat.

Another classic example is the cruise control system in a car, which regulates speed by adjusting the throttle. If the car slows down due to an uphill slope, the system detects the decrease in speed and increases engine power to compensate. Conversely, if the car speeds up when going downhill, the system reduces throttle to maintain the set speed.

Negative feedback is also widely observed in nature, such as in weather regulation. For instance, cloud formation plays a crucial role in balancing Earth's temperature. When surface temperatures rise, increased evaporation leads to more cloud cover, which in turn reflects sunlight away from the surface, reducing further heating. Conversely, when temperatures drop, reduced cloud formation allows more sunlight to reach the Earth's surface, helping to warm it up. In all these cases, the negative feedback mechanism ensures that deviations from the desired value are corrected automatically, enhancing stability and system performance.

Such kind of self-regulation feedback mechanisms are the most commonly seen control feature in engineering systems and natural phenonmena. For the remainder of this book, we will primarily focus on the analysis and design of negative feedback systems.

1.2.3 *The standard model*

For the reason that negative feedbacks are so common and important, we demonstrate a standard unit negative feedback model in Fig. 1.6. We will refer the model in Fig. 1.6 as the **base model** or the **standard model** for the rest of this book.

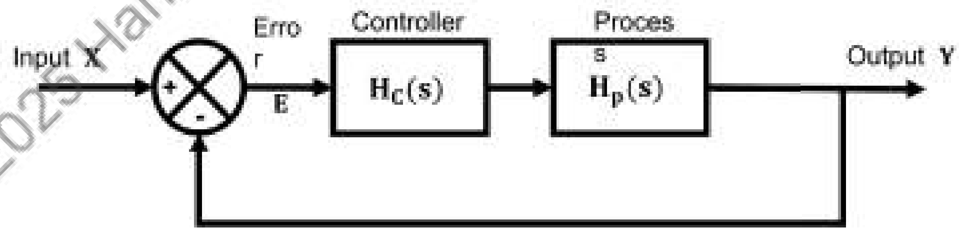


Figure 1.6: Our base model: negative unit feedback closed-loop model.

The **standard model** depicted in Fig. 1.6 represents a **standard negative unit feedback control loop**. In this framework, the **input signal** X is compared with the **output signal** Y at the summation block. The difference between these two signals forms the **error signal** E, which serves as a measure of the difference between the desired and actual system behavior.

This **error signal** is then processed by the **controller** H_c, which generates the appropriate control action to minimize the error. The controller's role is crucial as it determines how aggressively or smoothly the system responds to deviations from the desired state. The controller's output is then fed into the **process (plant)** H_p, which represents the system being controlled.

After passing through the process, the **output signal** Y is generated and subsequently fed back into the summation block for comparison, completing the **closed-loop system**. The feedback mechanism ensures that the system dynamically corrects itself by continuously adjusting the control input based on the observed output.

The time-domain computation of the output $y(t)$ is rather complicated comparing to open-loop systems thus it is omitted in this text. In later sections, we shall demonstrate a easier way to analyze the signals in control systems in the complex 's'-domain using Laplace transform.

This standard model provides a unified and simple framework for stability analysis, disturbance rejection, and performance enhancement, making negative feedback control an essential approach in engineering and control systems.

With feedback we can better:

- deal with system dynamics,
- be robust to uncertainty,
- realize a modular operation,
- gain more knowledge of the environment.

But feedback sometimes brings us trouble:

- increased system complexity,
- potentially affect the desired system characteristics like stability,
- potentially amplify noise and interference.

1.3 Response of a system

System dynamics can be modeled by differential equations. The complete solution of a differential equations is the sum of the homogeneous solution and the particular solution. The existence of these solutions suggests that:

- The *homogeneous solution* indicates that the system has an **natural response**. This natural response describes the physical characteristics of the system iteself based on the initial condition *without* any *input* added. Sometimes, the natural response is also called a zero-input response(ZIR).
- The *particular solution* indicates that the system has an **forced response** *dependent on the particular input* given *without any initial condition*. Sometimes, the natural response is also called a zero-state response(ZSR).

1.3 .1 Steady state and transient state of a response

When analyzing dynamic systems, we typically observe two main phases in the system response: the **transient state** and the **steady state**. These phases describe how the system behaves over time as it reacts to an input change.

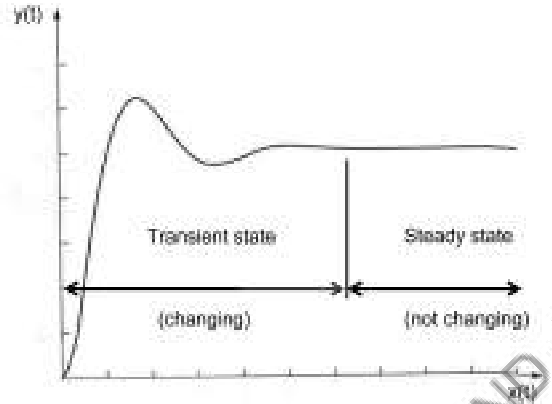


Figure 1.7: A visualization of transient state and steady state in the output time response plot.

The **transient state** refers to the initial phase of the system response, where the output undergoes relatively significant changes before reaching steady state. During this period, oscillations, overshoot, and damping effects are commonly observed as the system works to settle into its final value. The transient state is characterized by fluctuations, general trend of rising or falling, and even instability. Its duration depends on the system dynamics, including factors such as natural frequency, damping ratio, time constant, and the provided input.

As time progresses, the system transitions into the **steady state**, where the output stabilizes and ceases to exhibit significant changes. In this phase, the system reaches equilibrium, meaning any remaining variations in the output are minimal or due to small external disturbances. The steady state represents the long-term behavior of the system and is often the desired operating condition in control system design.

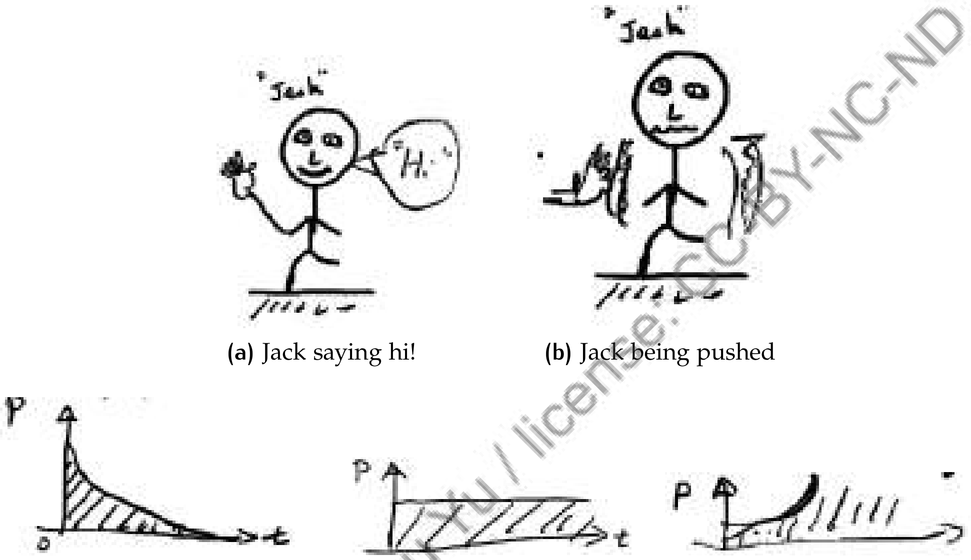
Figure 1.7 illustrates these two states. The transient response occurs at the beginning of the output signal, during which the system dynamically adjusts to a new condition. Once the output value gets close to the reference value from the input or stablize about a constant value, the system enters the steady-state phase, where the response remains constant or follows a predictable pattern based on the input reference.

To what extent can we determine that a system has entered the steady state? Must the system exactly match the input reference or converge to a single constant value? Or is a certain amount of error or fluctuation acceptable? We encourage the reader to reflect on these questions, as we will explore them in greater detail in a later section.

Understanding the distinctions between transient and steady states is crucial in control system analysis, as it allows engineers to analysis and evaluate system characteristics and overall performance. The goal in many control applications is to minimize the transient ripples and ensure a smooth and fast transition to the steady-state condition.

1.4 Stability and intuitions

Stability is a fundamental property of a control system that determines whether the system will return to equilibrium after being subjected to external disturbances. The illustrations in Fig. 1.8 provide an intuitive way to understand different stability scenarios through the analogy of Jack being pushed and how he responds to the disturbance.



(a) Jack saying hi! (b) Jack being pushed

(c) (d) (e)

Figure 1.8: Scenarios of Jack is being pushed.

In Fig. 1.8(a), Jack is saying hi while standing up straight on one foot, representing a system at equilibrium. In Fig. 1.8(b), a mysterious person is pushing Jack, exerting an external force on him. This symbolizes an external disturbance affecting the system. The lower illustrations, labeled (c), (d), and (e), depict how much power over time Jack needs to consume to counter the mysterious person.

- Fig. 1.8(c):
  If Jack is pushed and he gradually returns to his original position, the system is considered **stable**. This represents a SISO system where disturbances eventually decay to zero over time. In terms of power, the system initially expends energy to counteract the disturbance, but this energy dissipates over time, leading to a return to equilibrium. The shaded region indicates the transient response where power diminishes gradually.
  When Jack is being pushed, he initially consumes power to resist the

disturbance. Over time, the power consumption decreases as he regains balance and eventually stabilizes. This corresponds to an **stable system**, where the system naturally returns to equilibrium without requiring sustained effort. The total energy calculated by finding the integral of power with respect to time is finite. Jack only needs to consume some amount of energy to be able to balance himself again.

- Fig. 1.8(d):
  In this case, Jack does not return to his original state, but neither does he drift away uncontrollably. Instead, he maintains a new state or oscillates indefinitely with a constant amplitude. This also corresponds to a **stable system**. The shaded region indicates sustained power usage without decay.
  There exist some crucial differences comparing to the scenario in Fig. 1.8(c) where jack returns to the original status. The total energy calculated by finding the integral of power with respect to time is infinite because the power expended by Jack does not decay over time. Jack needs to consume an infinite amount of energy to be able to balance himself again which is not feasible at all. Jack will become tired after all. But in the meantime, the power required is finite and has an upperbound. That means that if, ideally, we have some constant supplement that can compensate the energy consumed by Jack, Jack can then sustain his balance!

- Fig. 1.8(e):
  If Jack is continuously pushed and he keeps losing balance, he needs to consume increasing amounts of power to counteract the growing disturbance. This reflects an **unstable system**, where the power required to maintain control do not have an upperbound and keeps rising, eventually leading to a **unbounded energy consumption** and **system failure**. Jack will fall into the ground, and perhaps even get injured.

These physical interpretations highlight how SISO control systems react to external influences. This example related to power shed light on two important factors on the nature of stability:

1. **finite energy**;
2. **bounded power**.

Obviously, having bounded power is a weaker constraint comparing to having finite energy. We shall discover more about how these constraints are incorporated into different notions of *stability*. A well-designed control system should ensures stability, preventing unbounded power usage and enabling predictable, controlled behavior in engineering applications.

1.5 Mathematical definition of stability

There exist a wide variety of notions and definitions of stability for control systems. We shall introduce some of the most widely used stability definitions in mathematical terms such that the reader can conduct the following analysis to systems described by ordinary differential equations.

1.5 .1 *Bounded-input bounded-output (BIBO) stability*

Definition 4 (Bounded-Input Bounded-Output (BIBO) Stability). A system is **BIBO stable** if, for every bounded input $x(t) = |u(t)| < \mathcal{M}$, the output remains bounded:

$$\sup_{t \geqslant 0} \|y(t)\|_\infty < \infty .$$

Recalling a general SISO system depicted in Eq. 1.1, in which time domain output is obtained by the convolution between the input and the system's inpulse response:

$$y(t) = \int_0^\infty x(\tau)h(t-\tau)\, d\tau$$

If we assume that x and h are both continuous, when we provide an bounded input x over $[0, \infty)$, y should also be bounded over $[0, \infty)$ for the system to be BIBO stable . Thus, the convolution integral needs to exist over the interval $t \in [0, \infty)$.

Note that $x(\tau)h(t-\tau)$ should be bounded for the convolution integral to be integrable. Thus, the system is BIBO stable if and only if h is bounded over the interval $t \in [0, \infty)$.

The above analysis leads to the essential condition for BIBO stability down to the following essential requirement :

For a *causal LTI* system with impulse response $h_\delta(t)$, the system is **BIBO stable** if the impulse response is **absolutely integrable**:

$$\int_0^\infty |h_\delta(t)| dt < \infty .$$

This ensures that the system's response does not grow unbounded for ***any*** finite input.

BIBO stability is the most commonly used stability definition in classical control systems. Unless specifically mentioned, we shall treat BIBO stability as our default stability notion.

1.5 .2 *Equilibrium points and system linearization*

Before we diverge into different defintions and types of stability, we need to know what is an equilibrium point.

Definition 5 (Equilibrium Point). Consider an arbitrary ordinary differential equation:

$$\frac{dy}{dt} = f(t, x(t)) .$$

A point $y_e \in \mathbb{R}$ is an equilibrium point if $f(y_e) = 0, \quad \forall t \in \mathbb{R}$.

An equilibrium point is a constant solution to an differential equation, other names are *stationary point*, *singular point*, *critical point*, and *rest point*. A constant solution indicates that if the system start from this equilibrium point it will rest on this equilibrium point.

With careful observation, it is convenient to deduce that if $f(y_e) = 0$ for an equilibrium point y_e we may use Taylor's Theorem to perform Taylor series expansion of $f(y)$ about the point $(y_e, f(y_e))$:

$$\begin{aligned} f(y) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(y_e)}{n!}(y - y_e)^n \\ &= f(y_e) + f'(y_e)(y - y_e) + \frac{f''(y_e)}{2!}(y - y_e)^2 + \frac{f'''(y_e)}{3!}(y - y_e)^3 + \\ &\quad \cdots + \frac{f^{(n)}(y_e)}{n!}(y - y_e)^n + \mathcal{O}((y - y_e)^{n+1}) \end{aligned}$$

This Taylor series expansion around an equilibrium point is particularly useful when we are handling an non-linear system. We can linearize the non-linear system by keeping the linear terms till the first order derivative. We can do this primarily due to the fact that as $y \to y_e$, higher-order exponential terms of $\|y - y_e\|$ decreases much much faster than the linear term $y - y_e$ such that they becomes negligible. For small perturbations around the equilibrium, the effect of higher-order terms are insignificant if $f'(y_e) \neq 0$.

Then we obtain an linearized approximation of $f(y)$ around the equilibrium point $(y_e, f(y_e))$. Putting $f(y_e) = 0$ in, the linearized system now becomes:

$$f(y) = f'(y_e)y .$$

After linearization, the system become trivial and the stability of the linearized system around that equilibrium point solely dependent on the derivative $f'(y_e)$ of $f(y)$ at the equilibrium point y_e.

- $f'(y_e) < 0$, the system can return to the equilibrium point,
- $f'(y_e) > 0$, the system cannot return to the equilibrium point.

An equilibrium point y_e is a logical choice to base the linearization technique when the system is suspected to oscillate around this equilibrium. Such that we may most likely to obtain a linearized function that minimizes the linearization error within acceptable torlerances. There are occassions that the Taylor series expansion is done about another point other than equilibrium but that would be very much system dependent. For now, by default, we perform liearization about an equilibrium point.

If the first derivative term at the equilibrium point $f'(y_e) = 0$, the quadratic term $f''(y_e)$ or the first non-zero term of lowest order becomes the dominant term. In this situation, it is meaningless to perform linearization at the equilibrium and ignore the higher-order terms. Meanwhile, it is also not a good idea to apply linearization if the pertubation is expected to be large, then linearization does not provide enough accurancy in the analysis and the linearization error becomes significant.

1.5 .3 Lyapunov stability

A system with an equilibrium point y_e is **Lyapunov stable** if, for every small perturbation ϵ, there exists a δ such that:

$$\|y(0) - y_e\| < \delta \quad \Rightarrow \quad \|x(t) - y_e\| < \epsilon, \quad \forall t \geqslant 0\,.$$

This means that if the system starts close to the equilibrium point, it remains arbitrarily close over time.

1.5 .4 Asymptotic stability

If $y(t)$ not only remains close but also **converges** to y_e as $t \to \infty$:

$$\lim_{t\to\infty} y(t) = y_e\,.$$

1.5 .5 Exponential stability

If the convergence occurs at an **exponential rate**, meaning there exist constants $c > 0$ and $\lambda > 0$ such that:

$$\|y(t) - y_e\| \leqslant c e^{-\lambda t} \|y(0) - y_e\|\,.$$

Exponential stability is a stronger condition than asymptotic stability.

1.5 .6 Unstable systems (Instability)

A system is **unstable** if its response grows unbounded over time. An unstable system cannot return to equilibrium without external control intervention.

An example of a unstable system is:

$$\dot{y}(t) = x(t)\,.$$

The differential equation above indicates that the system is an integrator. If we let $x(t)$ be a unit step signal, which is perfectly bounded. The output $y(t)$ becomes a ramp signal with slope equals to 1. Obviously, as t becomes larger, the output signal is going to continusouly increase at a constant rate and the output is unbounded.

1.5 .7 Marginal stability

A system is **marginally stable** or sometimes called critically stable if for some bounded input the output does not become unbounded over time but does not necessarily converge to an equilibrium either. Marginal stability is often used to fill the gap at the boundary between unstable systems and stable systems. In reality, the time response of a marginally stable system often appears to be an **undamped osciallation**.

With different inputs, a marginally stable system can also become unbounded (thus unstable) even if the input is bounded. This indicate that **a marginally stable system can not be BIBO stable**.

We utilize the example system with impulse response $h(t)$:

$$h(t) = \sin t$$

to demonstrate the statement above.

If the input $x_\delta(t)$ is an unit impulse function $\delta(t)$, the output $y_\delta(t)$:

$$\begin{aligned} y_\delta(t) &= [x_\delta * h](t) \\ &= [\delta * h](t) \\ &= h(t) = \sin t \end{aligned}$$

The output is bounded for an impulse input.

If the input $x_u(t)$ is an unit step function $u(t) = 1$, $t \in [0, \infty)$, the output $y_u(t)$:

$$\begin{aligned} y_u(t) &= [u * h](t) \\ &= \int_0^t \sin\tau \, d\tau \\ &= -\cos\tau|_0^t \\ &= 1 - \cos t \end{aligned}$$

The output is bounded for a step input.

If the input $x(t)$ is an sine function $\sin t$, the output $y(t)$:

$$
\begin{aligned}
y(t) &= \int_0^t h(\tau)x(t-\tau)\,d\tau \\
&= \int_0^t \sin\tau\sin(t-\tau)\,d\tau
\end{aligned}
$$

Since $\sin(t-\tau) = \sin t\cos\tau - \cos t\sin\tau$,

$$
= \sin t\int_0^t \sin\tau\cos\tau\,d\tau - \cos t\int_0^t \sin^2\tau\,d\tau
$$

$$
\begin{aligned}
\text{We know } \sin(2\tau) &= 2\sin\tau\cos\tau\,, \\
\text{and } \sin^2\tau &= \frac{1-\cos 2\tau}{2}\,,
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\sin t}{2}\int_0^t \sin 2\tau\,d\tau - \cos t\int_0^t \frac{1-\cos 2\tau}{2}\,d\tau \\
&= \frac{-\sin t\cos(2t)}{4} + \frac{\sin t}{4} - \frac{t\cos t}{2} + \frac{\cos t\sin(2t)}{4} \\
&= \frac{\cos t\sin(2t) - \sin t\cos(2t) + \sin t}{4} - \frac{t\cos t}{2} \\
&= \frac{-\sin(t-2t) + \sin t}{4} - \frac{t\cos t}{2} \\
&= \frac{\sin(t) - t\cos t}{2}
\end{aligned}
$$

Obviously, the $t\cos t$ in is unbounded. Hence the output $y(t)$ of the system becomes unbounded even if a bounded input is provided. The system is not BIBO stable.

1.6 Exercises

1. *(Safety first!)* Balance yourself on one foot with your eyes open for at least 30 seconds, then close your eyes and keep trying.

    a) How many time can you keep your balance with your eyes closed?

    b) Do you feel the difference?

    c) Describe the control system responsible for keeping you from falling down.

    d) Try to draw a block diagram. Explain what went wrong when you close your eyes.

2. Identify 5 control systems you encounter in daily life. Identify and draw the control loop, be specific.

3. Think about at least 5 factors in practice that will make a closed loop negative control system operate in an undesireable way?

4. Why is stability so important?

5. Are the following systems causal LTI systems?

    a) $\frac{dy(t)}{dt} = \frac{dx(t)}{dt} + x(t)$

    b) $\frac{dy(t)}{dt} + x(t) = \frac{d^2x(t)}{dt^2} + \frac{dx(t)}{dt} + x(t)$

    c) $\frac{dy(t)}{dt} = x(t)$

    d) $\frac{dy(t)}{dt} = t \cdot x(t)$

    e) $\frac{dy(t)}{dt} = x(t) + 2$

6. Are the natural response and forced response unrelated to each other? Why?

7. Why can we just simply add natural response and forced response for LTI systems?

8. Find the steady-state value of these time-domain output functions:

    a) $y = e^{-2t} + 1$

    b) $y = e^{-2t} + e^{t} - 10$